Linear combinations

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$

three 3-vectors.

Def (Linear combination): A linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an expression of the form:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

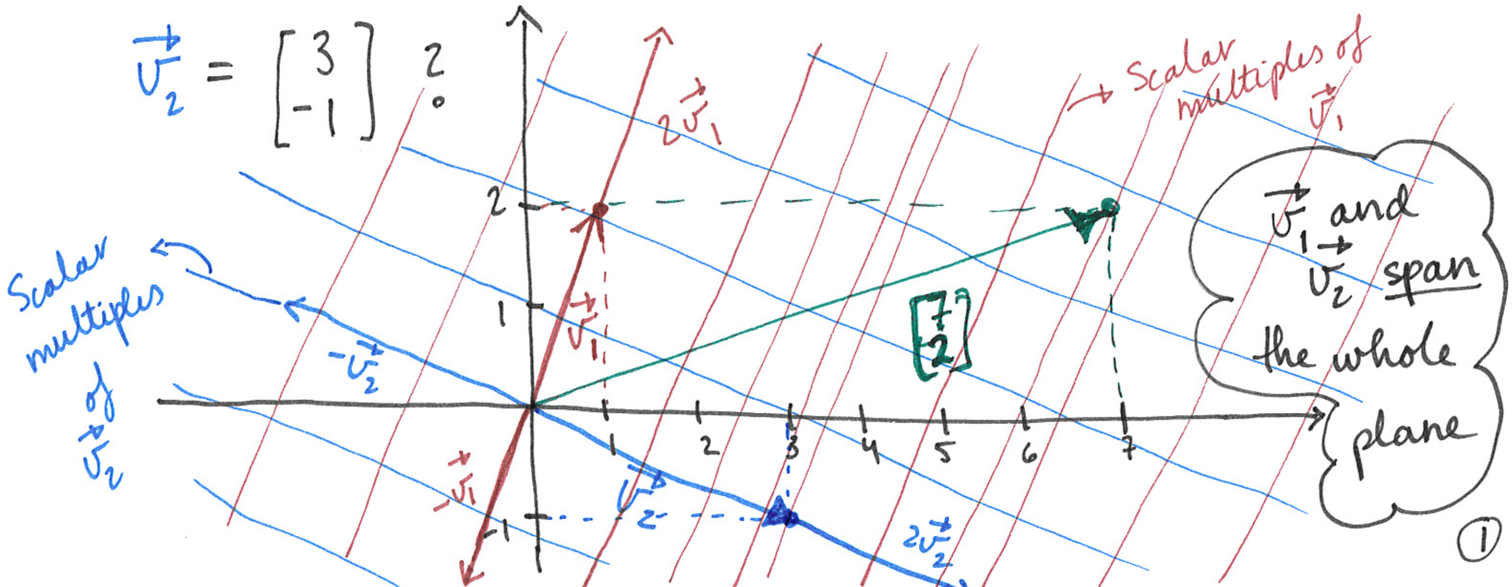
where c_1, c_2, c_3 are given numbers.

→ In general:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are m-vectors.

Ex: Is $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$?



Want to find c_1 and c_2 s.t.:

$$\underbrace{c_1 \vec{v}_1 + c_2 \vec{v}_2}_{\text{linear combination of } \vec{v}_1 \text{ and } \vec{v}_2} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

A vector equation

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 + 3c_2 &= 7 \\ 2c_1 - c_2 &= 2 \end{aligned}\right\}$$

$x + 3y = 7$
$2x - y = 2$

2x2 lin. syst.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 2 \end{array}\right] \xleftarrow{-2} \sim \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -12 \end{array}\right]$$

$$\begin{aligned} c_1 + 3c_2 &= 7 \\ -7c_2 &= -12 \end{aligned} \Rightarrow c_2 = \frac{12}{7} \approx 1.72$$

$$c_1 = 7 - 3c_2 = \dots = \frac{13}{7} \approx 1.86$$

$$(c_1, c_2) \approx (1.86, 1.72)$$

What does this mean?

$$1.86\begin{bmatrix}1\\2\end{bmatrix} + 1.72\begin{bmatrix}3\\-1\end{bmatrix} \approx \begin{bmatrix}7\\2\end{bmatrix}$$

$$\frac{13}{7}\begin{bmatrix}1\\2\end{bmatrix} + \frac{12}{7}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}7\\2\end{bmatrix}$$

Matrix multiplication

Def (Matrix multiplication):

the number of

If A, B matrices and # of columns A = # rows in B.

Then,

$$\underbrace{A \cdot B}_{\text{matrix multiplication of } A \text{ and } B} \text{ is defined.}$$

Ex:

$$\underset{A}{\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}} \cdot \underset{B}{\begin{bmatrix}3\\4\end{bmatrix}} = \begin{bmatrix}1\cdot 3 + 2\cdot 4\\2\cdot 3 + 1\cdot 4\end{bmatrix} = \begin{bmatrix}11\\10\end{bmatrix}$$

$$2 \times \underbrace{\cancel{2}}_{\substack{\#\text{ columns}\\ \text{in } A}} \cdot \underbrace{\cancel{2}}_{\substack{\#\text{ rows}\\ \text{in } B}} \times 1 = 2 \times 1$$

Matrix multiplication is defined

Why is the matrix product defined like this?

EQN. FORM : $\begin{cases} x + 2y = 5 \\ 2x + y = 1 \end{cases}$ $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$

$2 \times \cancel{2} \cdot \cancel{2} \times 1 = 2 \times 1$

SAME

$$\underbrace{A\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \vec{b}}_{\text{Matrix form}}$$

START : 13.05

Can take powers of square matrices:

Ex : $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$2 \times 2 \cdot 2 \times 2 = 2 \times 2$

$$= \begin{bmatrix} 1\cdot1 + 2\cdot2 & 1\cdot2 + 2\cdot1 \\ 2\cdot1 + 1\cdot2 & 2\cdot2 + 1\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

FORMULA for $A \cdot B$: If $A \cdot B$ is defined with $A = [a_{ij}]$, $B = [b_{ij}]$, then $A \cdot B = C = [c_{ij}]$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$

NOTE : $AB \neq BA$

Even though AB is defined, BA may not be

Ex :

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$2 \times 2 \cdot 2 \times 1 = 2 \times 1 \Rightarrow AB$ is defined.

SAME: MATRIX PROD. DEFINED

$$\overset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}$$

$2 \times 1 \cdot 2 \times 2 \Rightarrow BA$ is not defined.

NOT SAME: MATRIX PROD. NOT DEFINED

Linear systems

Ex:

$$\begin{aligned} x + y + z + w &= 4 \\ x - y \quad\quad + 2w &= 7 \\ 2x + 3y - z \quad\quad &= 10 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + y + z + w \\ x - y + 2w \\ 2x + 3y - z \end{bmatrix}$$

$$3 \times 4 \cdot 4 \times 1 = 3 \times 1$$

So the lin. syst. can be written: $= \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} = \vec{b}$

$$\boxed{A\vec{x} = \vec{b}}$$

The matrix form of the linear system

Matrix algebra

1) Add, subtract: $A + B$, $A - B$ (A and B are of same size)
2) Scalar multiplication: cA, c is a number (always defined)
3) Matrix multiplication: $A \cdot B$ (# columns in A = # rows in B)
4) Powers: A^n (A square matrix)

Ex:
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}_{2 \times 3,\ \text{not square}}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

$2 \times 3 \cdot 2 \times 3$ → Not defined

Special matrices

The identity matrix → Plays role of 1

$$I_2 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2}, \quad I_3 = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{3\times 3}, \ \dots$$

I_n → identity matrix of size n

→ We say that : $A^0 = I$

Property:

→ $A \cdot I = A$

Ex: $2\times 3 \cdot 3\times 3$

→ $I \cdot A = A$

Ex: $2\times 2 \cdot 2\times 3$

for any A, I of suitable dimension.

Ex:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\substack{A \\ 2\times 2}} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{I_2 \\ 2\times 2}} = \begin{bmatrix} 1\cdot 1+2\cdot 0 & 1\cdot 0+2\cdot 1 \\ 2\cdot 1+1\cdot 0 & 2\cdot 0+1\cdot 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}$$

DIY: $I_2 A = A$

The transpose

$$\underset{m \times n}{A} \xrightarrow[\boxed{\text{TRANSPOSE}}]{} \underset{n \times m}{A^T}$$

"A transpose" or "transpose of A"

Ex: $A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}$, $A^T = \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}}$

$A = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 7 & 3 \end{bmatrix}}$, $A^T = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 7 \\ 0 & 4 & 3 \end{bmatrix}}$

Main diagonal is preserved when transposing square matrices

Def: A is a <u>symmetric</u> matrix if

$$A = A^T$$

<u>Ex</u>: $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix} = B$

So $B = B^T$, hence B is symmetric.

Rules for matrix algebra

"Normal":

- $A + B = B + A$
- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $A \cdot (BC) = (AB) \cdot C$

"NOT normal"

- $AB \neq BA$
- $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

$$\neq A^2 + 2AB + B^2$$

(in general)

<u>Determinants</u>:

- $|A \cdot B| = |A| \cdot |B|$
- $|cA| = c^n |A|$ for A $n \times n$
- $|A^T| = |A|$

<u>Transpose</u>:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

All can be proved, but none shall