# Determinants and linear systems

EBA1180
Lect. 34
S25

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Cofactor expansion:

$$- 0 \begin{vmatrix} \cdots \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$- 0 \begin{vmatrix} \cdots \end{vmatrix}$$

$$= -0 \begin{vmatrix} \cdots \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0 \begin{vmatrix} \cdots \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix}$$

$$= -1(-1-1) - 1(1 \cdot (-1) - 1 \cdot 1)$$

$$= (-1) \cdot (-2) - 1 \cdot (-2) = 2 + 2 = \underline{\underline{4}}$$

# Alternative method for finding the determinant

1) Gaussian elimination until upper triangular matrix

   OBS: May be scaling and/or change of signs

2) Determinant is then the product of the diagonal elements.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

(-1)* row 1, add to row 3

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = E$$

(-1)* row 2, add to row 4

Upper triangular matrix

**RESULT:** If $E$ is an upper triangular matrix (i.e. all entries below main diagonal are 0), then $|E|$ is the product of diagonal entries.

NOTE: All echelon forms are upper triangular.

Ex ctd: i) $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2)$

E upper triangular: Result

$= \underline{\underline{4}}$

ii) NB: Here $|A| = |E| = 4$

Why is the result true?

Ex: 

Cofactor expansion

$$|E| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$- 0 \begin{vmatrix} \cdots \end{vmatrix} + 0 \begin{vmatrix} \cdots \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix}$$

$$= 1 \cdot 1 \cdot \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 1 \cdot 1 \cdot \left( (-2) \cdot (-2) - 0 \cdot 0 \right)$$

$$= 1 \cdot 1 \cdot (-2) \cdot (-2) = 4$$

Pattern holds for any upper triangular matrix.

Ex:
$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 2 \cdot 0 = 0$$

UPPER TRIANGULAR

Q: How many solutions of corresp. lin. system?

0 or $\infty$ many because $|\cdot| = 0$

START 11.00

RESULT

(Change in determinant from elementary row operations)

If $A \sim B$ via elementary row operations, then

i) Switch two rows $\Rightarrow |B| = -|A|$

ii) Multiply a row by $c \neq 0 \Rightarrow |B| = c|A|$

iii) Add a multiple of a row to another row:
$$|B| = |A|$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $|A| = 0 \cdot 1 - 1 \cdot 1 = \underline{-1}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$$

Switch row 1 and 2

$|B| = 1 \cdot 1 = \underline{1} = -|A|$

Recall: RESULT: i) $|A| \neq 0 \Rightarrow$ One solution

ii) $|A| = 0 \Rightarrow \infty$ many OR no solutions

Ex:

$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}\right\} (*)$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$= 2 \neq 0$, so $|A| \neq 0 \Rightarrow$ RESULT

$(*)$ has one unique solution.

Check via row reduction

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \begin{matrix} \\ \\ -2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Pivot in each diagonal element:
One unique solution

Ex alt.:

$$x + y + z = 3$$
$$x + 2y + 4z = 7$$
$$2x + 3y + 5z = 10$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 10 - 12 - (5 - 8) + 3 - 4$$

$$= -2 + 3 - 1 = \underline{0}$$, so $|A| = 0 \Rightarrow$ RESULT

No solutions or infinitely many solutions.

Which one?

## Check via row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 5 & 10 \end{array}\right] \begin{array}{l} \\ -1 \\ -2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array}\right] \begin{array}{l} \\ \\ -1 \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \infty \text{ many solutions}$$

No pivot in column 3 $\Rightarrow$ $z$ is free

## Linear systems with parameters

Ex:

$$x + y = 4$$
$$x + ay = 6$$

$x, y$ : variables — endogeneous

$a$ : parameter — exogeneous

Solution depends on the parameter

Solve: Gaussian elimination

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a & 6 \end{array}\right] \begin{array}{l} \\ -1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

$a-1$: Pivot or not?

DEPENDS!

Two cases:

$a-1=0$

$\boxed{a=1}$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right]$$

$\Downarrow$

No solutions!

$a \neq 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

Pivot!

$x + y = 4$

$(a-1)y = 2 \Rightarrow y = \dfrac{2}{a-1}$

$$x = 4 - y = 4 - \frac{2}{a-1} = \frac{4a-4-2}{a-1} = \frac{4a-6}{a-1}$$

Solution:

$(x,y) = \left( \dfrac{4a-6}{a-1}, \dfrac{2}{a-1} \right)$, $a \neq 1$

One unique solution!