

Determinants and linear systems

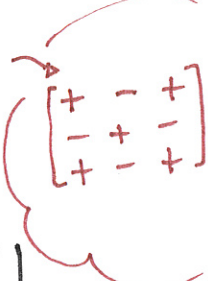
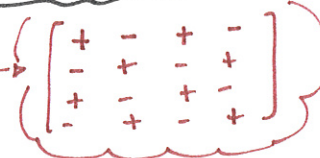
ERA1180

Lect. 34

S25

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$



Cofactor expansion:

$$- 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$- 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix}$$

$$= - 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix}$$

$$- 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix} - 0 \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix}$$

$$= -1(-1-1) - 1(1 \cdot (-1) - 1 \cdot 1)$$

$$= (-1) \cdot (-2) - 1 \cdot (-2) = 2 + 2 = \underline{\underline{4}}$$

Alternative method for finding the determinant

- 1) Gaussian elimination until upper triangular matrix
↳ OBS: May be scaling and/or change of signs
- 2) Determinant is then the product of the diagonal elements.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

(-1) row 1, add to row 3*

\sim
(-1) row 2, add to row 4*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = E$$

Upper triangular matrix

RESULT: If E is an upper triangular matrix (i.e. all entries below main diagonal are 0), then $|E|$ is the product of diagonal entries.

NOTE: All echelon forms are upper triangular.

Ex ctd: i) $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2)$
 E upper triangular;
Result $= \underline{\underline{4}}$

ii) NB: Here $|A| = |E| = 4$

Why is the result true?

Ex: $|E| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$ (Cofactor expansion)

$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

$- 0 \mid \dots \mid + 0 \mid \dots \mid - 0 \mid \dots \mid$

$= 1 \cdot 1 \cdot \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 1 \cdot 1 \cdot ((-2) \cdot (-2) - 0 \cdot 0)$

$= 1 \cdot 1 \cdot (-2) \cdot (-2) = 4$

Pattern holds for any upper triangular matrix.

Ex:
$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 2 \cdot 0 = 0$$

UPPER TRIANGULAR

Q: How many solutions of corresp. lin. system?

0 or ∞ many because $1 \cdot 1 = 0$

START
11.00

RESULT

(Change in determinant from elementary row operations)

If $A \sim B$ via elementary row operations, then

i) Switch two rows $\Rightarrow |B| = -|A|$

ii) Multiply a row by $c \neq 0 \Rightarrow |B| = c|A|$

iii) Add a multiple of a row to another row:

$$|B| = |A|$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, |A| = 0 \cdot 1 - 1 \cdot 1 = \underline{-1}$

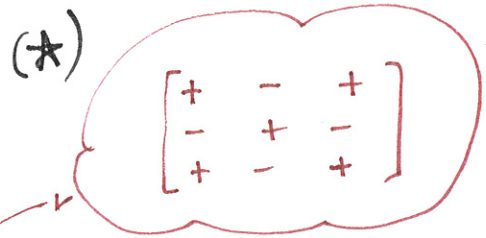
$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$

Switch row 1 and 2

$$|B| = 1 \cdot 1 = \underline{1} = -|A|$$

Recall: **RESULT:** i) $|A| \neq 0 \Rightarrow$ One solution
 ii) $|A| = 0 \Rightarrow$ ∞ many OR no solutions

Ex:
$$\left. \begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned} \right\}$$



$|A|$ =
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$$= 2 \neq 0, \text{ so } |A| \neq 0 \Rightarrow$$

RESULT

(*) has one unique solution.

Check via row reduction

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{matrix} \leftarrow -1 \\ \leftarrow -1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \begin{matrix} \leftarrow -2 \end{matrix}$$

$$\sim \begin{bmatrix} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & 3 \\ 0 & 0 & \textcircled{2} \end{bmatrix}$$

Pivot in each diagonal element:
One unique solution

Ex alt.:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ 2x + 3y + 5z &= 10 \end{aligned}$$

$$\begin{aligned} \underline{|A|} &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ &\quad - 1 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \\ &\quad + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{aligned}$$

$$= 10 - 12 - (5 - 8) + 3 - 4$$

$$= -2 + 3 - 1 = \underline{0}, \text{ so } |A| = 0 \Rightarrow$$

No solutions or infinitely many solutions.

Which one?

Check via row reduction:

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 5 & 10 \end{array} \right] \xrightarrow{\substack{-1 \\ -2}} \sim \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 0 & \textcircled{1} & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array} \right] \xrightarrow{-1} \sim \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 3 \\ 0 & \textcircled{1} & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\begin{matrix} x & y & z \\ \uparrow & & \uparrow \\ \text{No pivot} & & \text{in column 3} \Rightarrow z \text{ is free} \end{matrix}$

$\Rightarrow \infty$ many solutions

Linear systems with parameters

Ex: $x + y = 4$
 $x + ay = 6$

x, y : variables

a : parameter

Solve: Gaussian elimination

$$\left[\begin{array}{c|c} \textcircled{1} & 1 \\ 1 & a \end{array} \middle| \begin{array}{c} 4 \\ 6 \end{array} \right] \xrightarrow{-1} \sim \left[\begin{array}{c|c} \textcircled{1} & 1 \\ 0 & \textcircled{a-1} \end{array} \middle| \begin{array}{c} 4 \\ 2 \end{array} \right]$$

exogeneous

Solution

depends on the parameter

Pivot or not?

DEPENDS!

Two cases:

$a-1=0$

$a=1$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array} \right]$$

↓
No solutions!

$a \neq 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array} \right]$$

Pivot!

$$\begin{aligned} x + y &= 4 \\ (a-1)y &= 2 \Rightarrow y = \frac{2}{a-1} \\ x &= 4 - y = 4 - \frac{2}{a-1} \\ &= \frac{4a - 4 - 2}{a-1} \\ &= \frac{4a - 6}{a-1} \end{aligned}$$

Solution:

$$(x, y) = \left(\frac{4a-6}{a-1}, \frac{2}{a-1} \right),$$

One unique solution!

$a \neq 1$