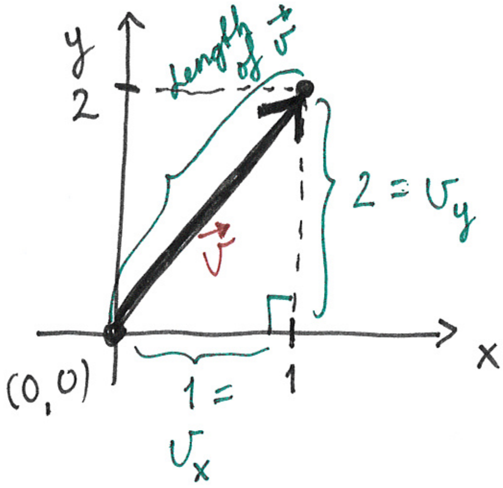
Geometric interpretation of vectors

EBA 1180
Lect. 33
S25

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

corresp. to an arrow from $(0,0)$ to $(v_x, v_y) = (1,2)$



Pythagorean theorem:

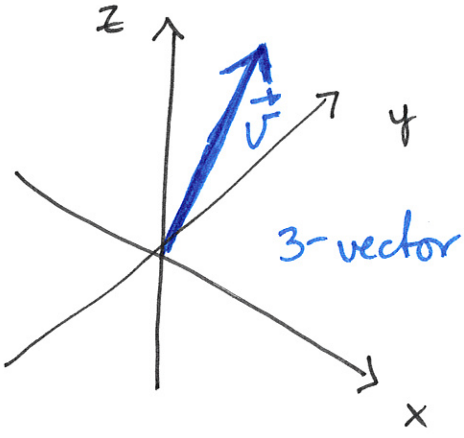
$$v_x^2 + v_y^2 = \left(\text{length of } \vec{v}\right)^2$$

$$\text{length of } \vec{v} = \sqrt{v_x^2 + v_y^2}$$

A vector has length and direction.

$$\|\vec{v}\| = \left\| \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right\| = \sqrt{v_x^2 + v_y^2}$$

Ex: $\|\vec{v}\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$



Def: (Length of vector)

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

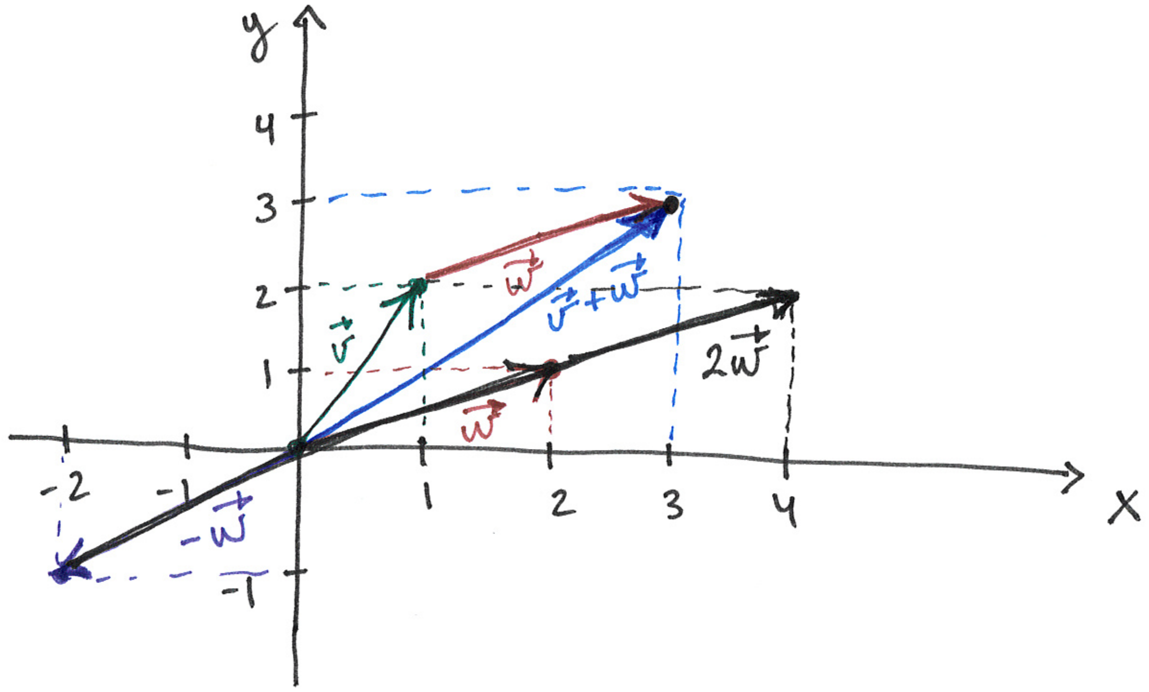
Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ADD: $\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

SCALAR MULTIPLICATION: $2\vec{w} = 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$(-1)\vec{w} = -\vec{w} = (-1) \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Visualize vector operations:



Determinants

"the determinant of A"

$$A \longrightarrow \overbrace{\det}(A) = \overbrace{|A|}$$

n × n matrix

Square!

a number

Ex:

$$A = \underset{2 \times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}, \quad \det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2 \cdot 2 - 1 \cdot 1$$

$$= \underline{\underline{3}}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

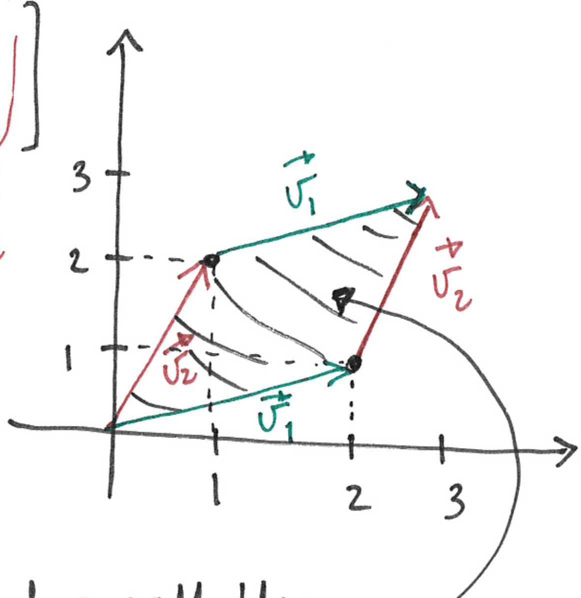
FORMULA (Determinant, n = 2)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Interpretation:

$$A = \left[\underset{\vec{v}_1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \quad \underset{\vec{v}_2}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \right]$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



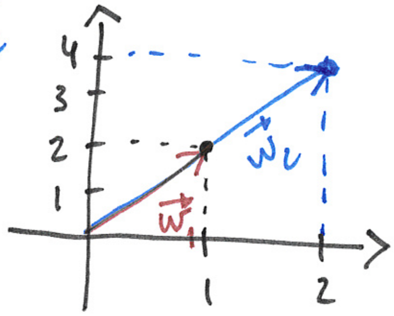
$$\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \underline{3}$$

Can prove: $\left| \det(A) \right| =$ area of parallellogram spanned by $\vec{v}_1$ and $\vec{v}_2$

NOTE: $\begin{vmatrix} \vec{v}_2 & \vec{v}_1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1\cdot 1 - 2\cdot 2 = \underline{-3}$

Area can't be neg.: Need to take abs. value

Ex: $$\begin{array}{c} \left|\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array}\right| \\ \vec{w}_1 \quad \vec{w}_2 \end{array} = 1\cdot 4 - 2\cdot 2 = 0$$



Parallelogram spanned by $\vec{w}_1$ and $\vec{w}_2$:

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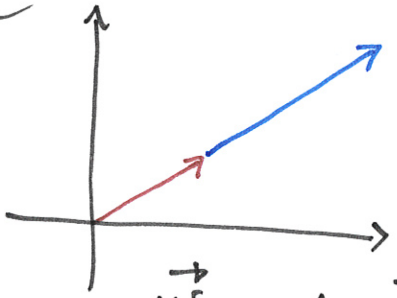
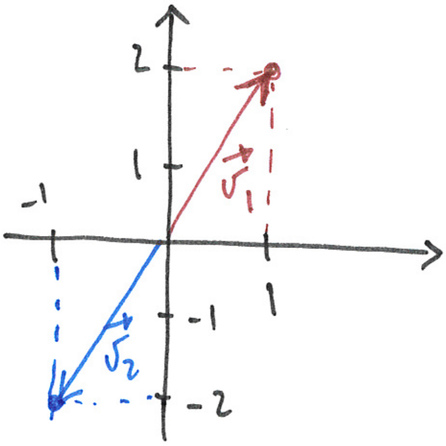


Area = 0

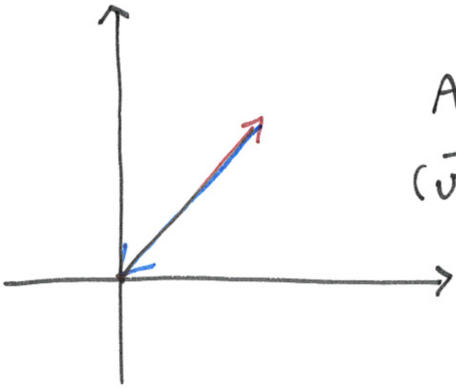
$\vec{w}_1$ and $\vec{w}_2$ are scalar multiples

Ex: $$\begin{array}{c} \left|\begin{array}{cc} 1 & -1 \\ 2 & -2 \end{array}\right| \\ \vec{v}_1 \quad \vec{v}_2 \end{array} = 1\cdot(-2) - (-1)\cdot 2$$

$$= 0$$



Parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2$:



Area = 0

($\vec{v}_1$ and $\vec{v}_2$ are scalar multiples)

Result: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ satisfy:

The vectors are scalar multiples

The general case, $n \times n$: METHOD for finding determinants

Cofactor expansion

A, $n \times n$ matrix

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$, 3×3 square

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \underset{\substack{\text{cofactor} \\ \text{expansion} \\ \text{along row} \\ 1}}{=} 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$$

where C_{11}, C_{12}, C_{13} are cofactors:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where $\overset{\text{minor}}{M_{ij}}$ is the determinant of the submatrix you get when you delete row i, column j.

So:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1 \cdot (-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix}$$

$$+ 1 \cdot 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$$= \underline{\underline{2}}$$

NB: • Cofactor expansion along any row/column gives same result.

- Any determinant can be computed via cofactor expansion.
- Exploit the 0's ! Use rows/columns with lots of zeros to simplify computations.

Connection between # solutions of linear systems and determinants

→ Start with $n \times n$ lin. syst (# eqns = # variables)

Then, corresp. coefficient matrix is an $n \times n$ matrix

Ex:
$$\underbrace{\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ x+3y+9z&=13 \end{aligned}}_{3\times 3 \text{ linear system}}$$

write as: $A\vec{x} = \vec{b}$ (matrix form)

where
$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{\text{coefficient matrix}}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}$$

Result: $\boxed{A\ n\times n}$

i) $|A| \neq 0 \Rightarrow$ One unique solution.

ii) $|A| = 0 \Rightarrow$ No solutions or infinitely many solutions

Theorem : (Cramer's rule)

Consider a linear system, $A\vec{x} = \vec{b}$, with coefficient matrix A and r.h.s. $\vec{b}$, such that A is square $(n \times n)$ with $|A| \neq 0$. Then, the solution of the linear system is:

$$x_1 = \frac{|A_1(\vec{b})|}{|A|}, \quad x_2 = \frac{|A_2(\vec{b})|}{|A|}, \ldots,$$

$$x_n = \frac{|A_n(\vec{b})|}{|A|}$$

where $A_i(\vec{b})$ is the matrix you get when you replace the i'th column in A with $\vec{b}$.