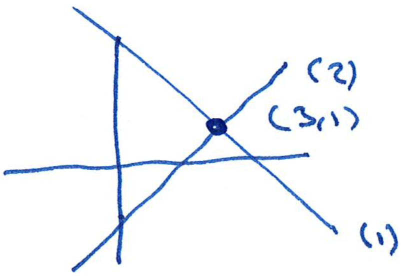
Plan

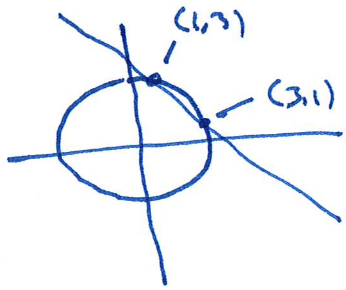
1 Systems of equations

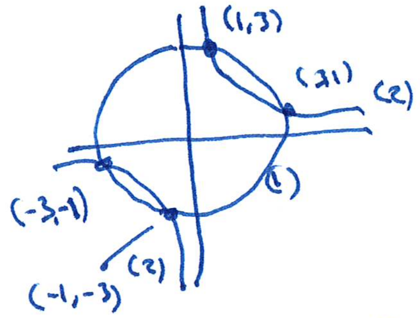
2 Linear systems and Gaussian elimination

① Systems of equations

Ex: ① $x+y=4$ (1), $x-y=2$ (2) ② $x+y=4$ (1), $x^2+y^2=10$ (2) ③ $x^2+y^2=10$ (1), $xy=3$ (2)







$x+y=4 \Rightarrow y=4-x$

$x-y=2 \Rightarrow y=x-2$

"Expected" number of solutions $= d_1 \cdot d_2$

$d_1 =$ degree of first equ.

$d_2 =$ — " — second "

Substitution:

(1) $x+y=4$

$\underline{y=4-x}$

(2) $x^2+(4-x)^2=10$

$x^2+16-8x+x^2=10$

$2x^2-8x+6=0 \quad |:2$

$x^2-4x+3=0$

$\underline{x=1}, \ \underline{x=3}$

$y=3 \quad y=1$

$(x,y)=(1,3),(3,1)$

$xy=3 \Rightarrow y=3/x$

Substitution:

(2) $\underline{y=3/x}$

(1) $x^2+(3/x)^2=10$

$\cdot x^2 \,|\, x^2+9/x^2=10$

$x^4+9=10x^2$

$x^4-10x^2+9=0$

$u=x^2$: $u^2-10u+9=0$

$u=1$ or $u=9$

$x^2=1$ " $x^2=9$

$x=\pm 1$ " $x=\pm 3$

$(x,y)=(1,3),(-1,-3),$

$(3,1),(-3,-1)$

2 Linear systems = linear systems of equations

Ex:

$$x+y+z=3$$
$$x+2y+4z=7$$
$$x+3y+9z=13$$

3×3 linear system ← all eqns. are linear

\# equations = 3

\# variables = 3

in standard form

Defn An m×n linear system is a system of m equations in n variables where each equation is linear.

In standard form:

(variables $x_1, x_2, \ldots, x_n$)

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}}$$

Regarding linear equations:

$$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = b$$

(degenerate case, linear eqn.)

$b \neq 0$: no solutions

$b = 0$: all $(x_1, x_2, \ldots, x_n)$ are solutions

Method: Gaussian elimination – general method for solving any linear system

Ex:

$$\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ x+3y+9z&=13 \end{aligned} \rightsquigarrow \begin{array}{cccc} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 4 & 7\\ 1 & 3 & 9 & 13 \end{array}\right) \\ \uparrow\ \ \ \uparrow\ \ \ \uparrow \\ x\text{ col.}\ \ y\text{ col.}\ \ z\text{ col.} \end{array}$$

3x3 linear system in std form

augmented/extended coeff. matrix of the lin. system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 4 & 7\\ 1 & 3 & 9 & 13 \end{array}\right) \quad (-1\ -1\ -1\,|\,-3)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 1 & 3 & 9 & 13 \end{array}\right) \quad (-1\ -1\ -1\,|\,-3)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 2 & 8 & 10 \end{array}\right) \quad (0\ -2\ -6\,|\,-8)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 3 & 4\\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

Elementary row operations:

i) switch two rows
ii) multiply a row by $c \neq 0$
iii) add a multiple of one row to another row

– preserve solutions
– all we need!

pivot = first non-zero entry in a row

Echelon form

A matrix is in echelon form if:

i) zero rows are in the bottom of the matrix
ii) each pivot is further to the right than the pivots above

Fact: Any matrix can be transformed to an echelon form using elementary row op.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \end{array} \rightsquigarrow \begin{aligned} x + y + z &= 3 \quad (1) \\ y + 3z &= 4 \quad (2) \\ 2z &= 2 \quad (3) \end{aligned}$$

echelon form

Back substitution:

(3) $\frac{2z}{2} = \frac{2}{2}$ $\quad z = 1$

(2) $y + 3z = 4$

$y + 3 \cdot 1 = 4$

$y = 1$

(1) $x + y + z = 3$

$x + 1 + 1 = 3$

$x = 1$

Solutions

$(x, y, z) = (1, 1, 1)$

In general:

Start from the last eqn. and solve for the variable corresponding to a pivot, then move to the next, substitute and repeat.

Ex:

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x + 2y + z + w = 2 \quad (1)$

$y = 4 \quad (2)$

$\cancel{0 \cdot x + 0 \cdot y + 0 \cdot z + 0 \cdot w = 0}$

degenerate $\Rightarrow$ always satisfied

(2) $y = 4$

(1) $x + 2y + z + w = 2$

$x + 2 \cdot 4 + z + w = 2$

$x = -6 - z - w$

z, w are free variables

$x = -6 - z - w = -6 - s - t$

$y = 4$

$z = s$

$w = t$

Solutions:

$(x, y, z, w) = (-6 - s - t, 4, s, t)$

where s, t are parameters

(z, w are free variables)

infinitely many solutions

What if we instead had:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 5 \end{array}\right)$$

nondegenerate case with no solutions