More integration: Applications of integration

EBA 1180
Lect. 6
(30)
Spring 25

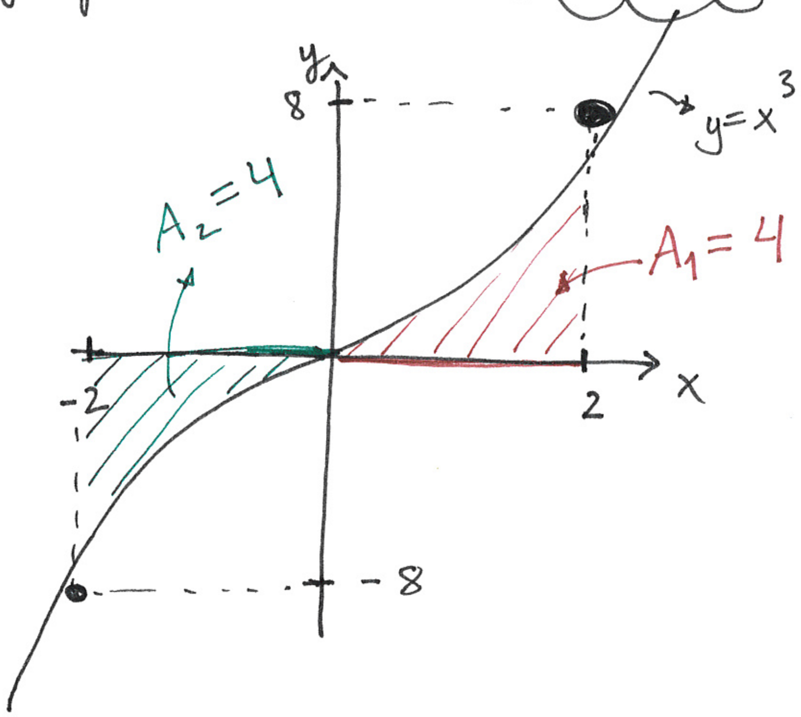
What about integration of function that are not always ≥ 0?

Ex: $\int_{-2}^{2} x^3\,dx$

$= \left[\frac{1}{4}x^4\right]_{x=-2}^{2}$

$= \frac{1}{4}\left(2^4 - (-2)^4\right)$

$= \frac{1}{4}(16-16) = 0 \rightarrow$ Why? Split!



$I_1 = \int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=0}^{2} = \frac{1}{4}(\overbrace{2^4}^{16} - 0^4)$

$= \underline{4}$

$I_2 = \int_{-2}^{0} x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=-2}^{0} = \frac{1}{4}(0^4 - \overbrace{(-2)^4}^{16})$

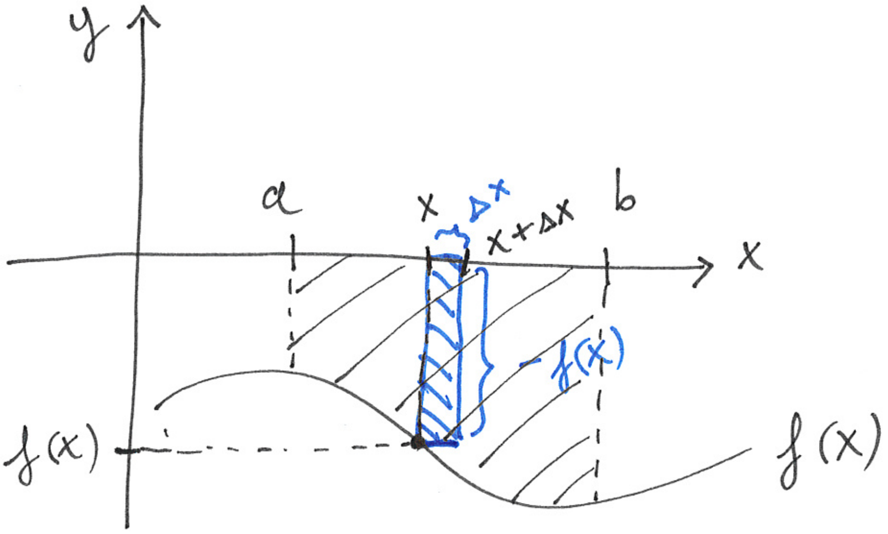
$= \frac{-16}{4} = \underline{-4}$

So: $\underbrace{I_1 + I_2}_{A_1 - A_2} = 4 + (-4) = 0$

When $f(x) \leq 0$ in $[a, b]$:

Area between the x-axis and the graph of $y = f(x)$ in $[a,b]$ is:

$$\underbrace{A}_{\text{area}} = \int_a^b -f(x)\,dx, \quad \text{so} \quad \int_a^b f(x)\,dx = -A$$



Height of rectangle:

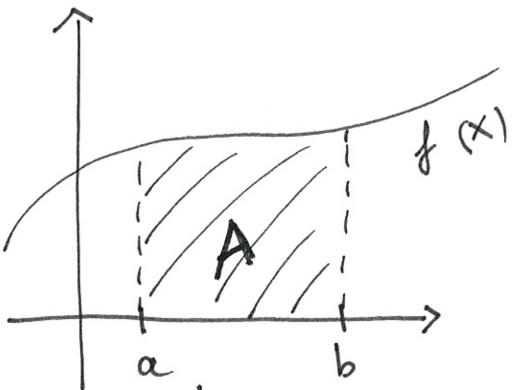
$0 - f(x) = -f(x)$

a positive number

$$\text{area of rectangle} = \underbrace{-f(x)}_{\text{height}}\,\Delta x$$

3 cases:

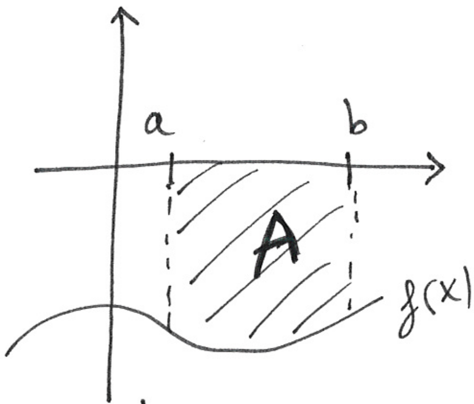
i) $f(x) \geq 0$:



$$A = \int_a^b f(x)\,dx$$

(as before)

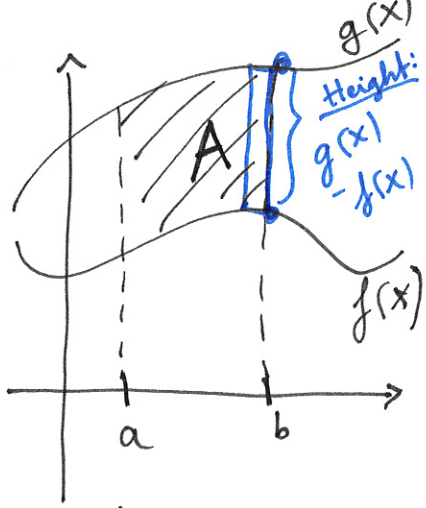
ii) $f(x) \leq 0$:



$$A = \int_a^b -f(x)\,dx$$

$$-A = \int_a^b f(x)\,dx$$

iii) $f(x) \leq g(x)$



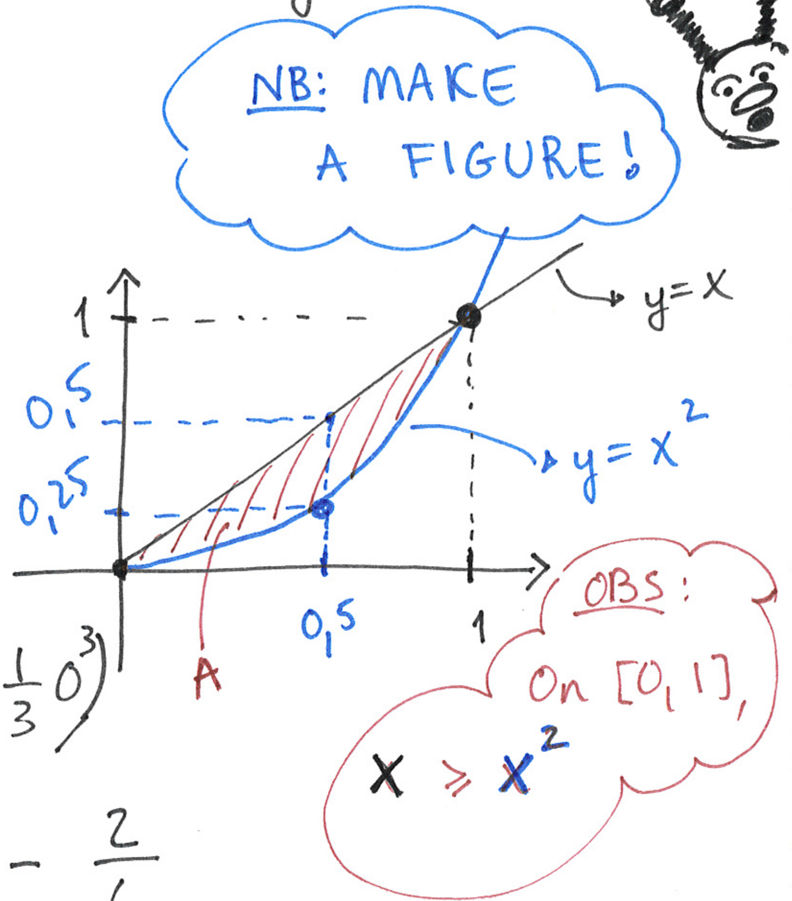
$$A = \int_a^b (g(x) - f(x))\,dx$$

NB: An area is non-negative

Ex: What is the area between $y = x$ and $y = x^2$ in $[0,1]$?

$$A = \int_0^1 x - x^2 \, dx$$

iii)



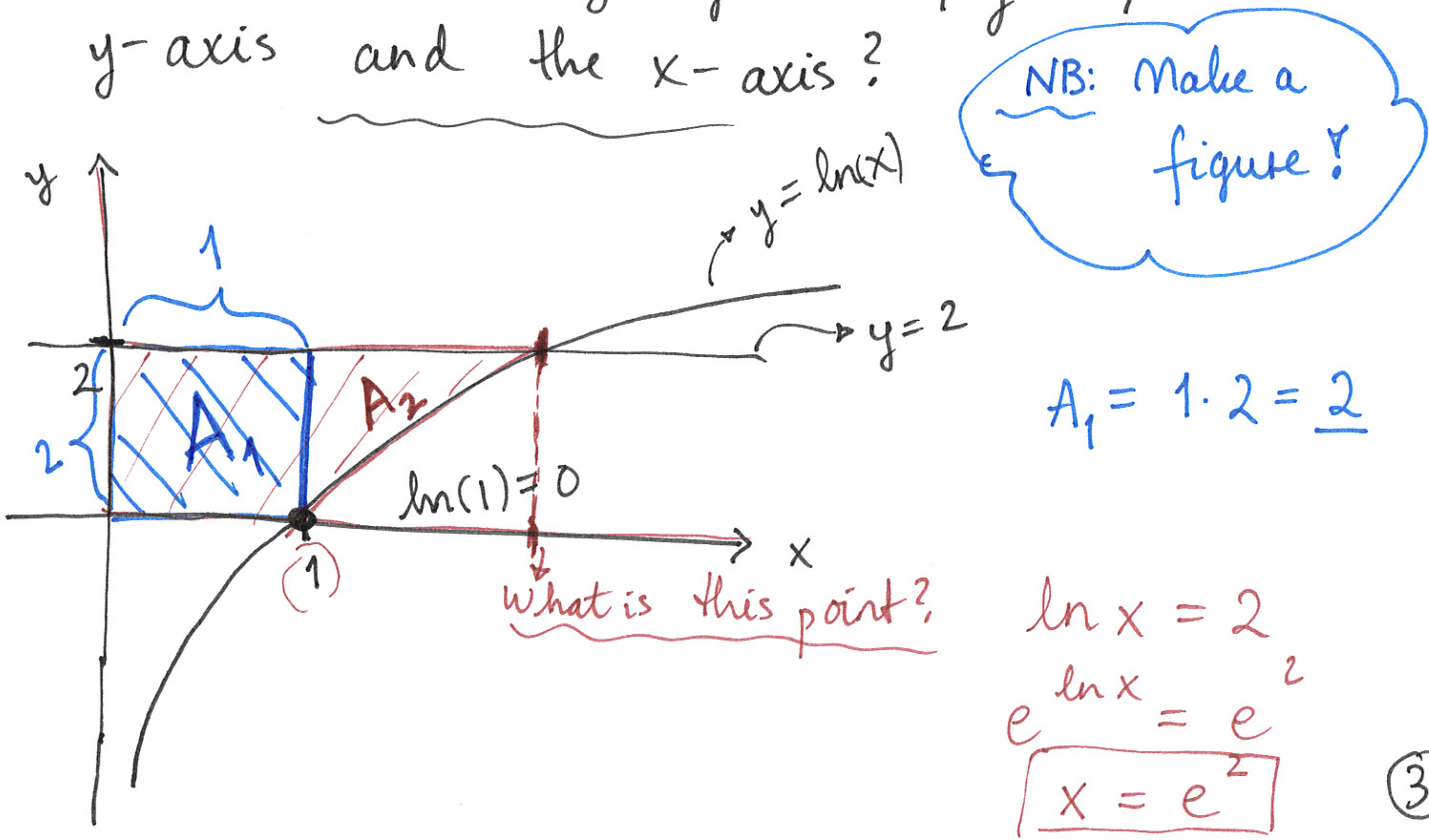
$$= \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_{x=0}^{1}$$

$$= \frac{1}{2} 1^2 - \frac{1}{3} 1^2 - \left(\frac{1}{2} 0^2 - \frac{1}{3} 0^3 \right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6}$$

$$= \underline{\underline{\frac{1}{6}}} \quad (\approx 0{,}167)$$

Ex: Area bounded by $y = \ln x$, $y = 2$, the y-axis and the x-axis?



$$A_1 = 1 \cdot 2 = \underline{2}$$

$$\ln x = 2$$

$$e^{\ln x} = e^2$$

$$\boxed{x = e^2}$$

$$A_2 = \int_1^{e^2} 2 - \ln x \, dx$$

$$\text{Area} = A_1 + A_2 = 2 + \int_1^{e^2} 2 - \ln x \, dx$$

int by parts: $\int \ln x = 1 \cdot \ln x$

$$= 2 + \Big[\underbrace{2x - (x\ln x - x)}_{\text{int by parts}} \Big]_{x=1}^{e^2}$$

$$= 2 + \Big[3x - x\ln x \Big]_{x=1}^{e^2}$$

$$= 2 + \big(3e^2 - e^2 \overbrace{\ln(e^2)}^{2}\big) - \big(3 \cdot 1 - 1 \overbrace{\ln(1)}^{0}\big)$$

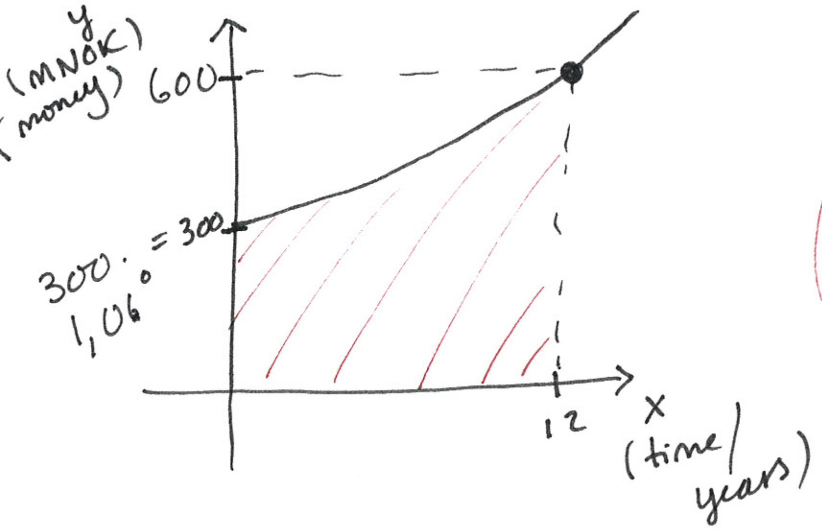
$$= 2 + 3e^2 - 2e^2 - 3 = \underline{\underline{e^2 - 1}} \; (\approx 6{,}389)$$

START 13.02

Economic applications of the definite integral

Continuous cash flows

Ex: $f(x) = 300 \cdot 1{,}06^x$ (cash flow in MNOK/year)



Rule of 72: Doubling takes approx. $\frac{72}{6} = 12$ years when money grows with 6% per year

Total cash flow in 12 years = the area under the graph i [0, 12]

$$= \int_0^{12} f(x)\,dx = \int_0^{12} 300 \cdot 1{,}06^x \, dx$$

$\int a^x dx$

Formula from 1st integration lecture

$$= \left[300 \, \frac{1{,}06^x}{\ln(1{,}06)} \right]_{x=0}^{12}$$

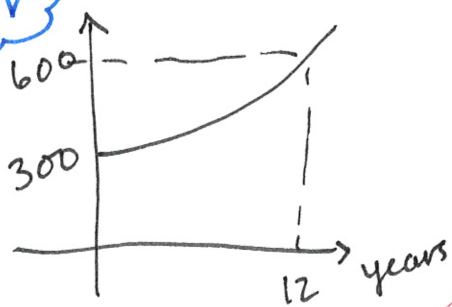
$$= \frac{300}{\ln(1{,}06)} \left[1{,}06^x \right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1{,}06)} \left(1{,}06^{12} - \underbrace{1{,}06^0}_{1} \right)$$

$$= \frac{300}{\ln(1{,}06)} \left(1{,}06^{12} - 1 \right) \approx \underline{5\,211 \text{ MNOK}}$$

Net present value of a continuous cash flow

NPV



$f(x) = 300 \cdot 1{,}06^x$, cash flow

ASSUME: r = discount rate = 10%, continuous discounting.

Money in the future is worth less than money today: Inflation

NPV:

$$\int_0^{12} \underbrace{f(x)}_{\text{cash flow}} e^{-rx}\,dx = \int_0^{12} \overbrace{300\cdot 1{,}06^x}^{f(x)}\, e^{\overbrace{-0{,}10x}^{\text{discounting}}}\,dx$$

$$= 300\int_0^{12} \underbrace{1{,}06^x}_{=} e^{-0{,}10x}\,dx$$

$$= 300\int_0^{12} e^{\ln(1{,}06)x}\, e^{-0{,}10x}\,dx$$

$$e^{\ln(1{,}06)x} = \left(e^{\ln(1{,}06)}\right)^x = (1{,}06)^x = 1{,}06^x$$

$$= 300\int_0^{12} e^{(\ln(1{,}06)-0{,}10)x}\,dx$$

Substitution:

$$u = (\ln(1{,}06)-0{,}10)x$$

$$du = (\ln(1{,}06)-0{,}10)\,dx$$

$$\int e^{(\ln(1{,}06)-0{,}10)x}\,dx = \int e^u \frac{1}{\ln(1{,}06)-0{,}10}\,du = \frac{1}{\ln(1{,}06)-0{,}10} e^u + C$$

$$= 300\left[\frac{1}{\ln(1{,}06)-0{,}1} e^{(\ln\ldots)x}\right]_{x=0}^{12}$$

$\ln(1{,}06)-0{,}1$

$$= \frac{300}{\ln(1{,}06)-0{,}1}\left(e^{(\ln(1{,}06)-0{,}1)\cdot 12} - \underbrace{e^0}_{1}\right)$$

$$\approx \underline{\underline{2832}} \text{ MNOK}$$

FORMULAS (Economic applications of definite integrals)

Total cash flow: $\int_0^T f(x)\,dx$

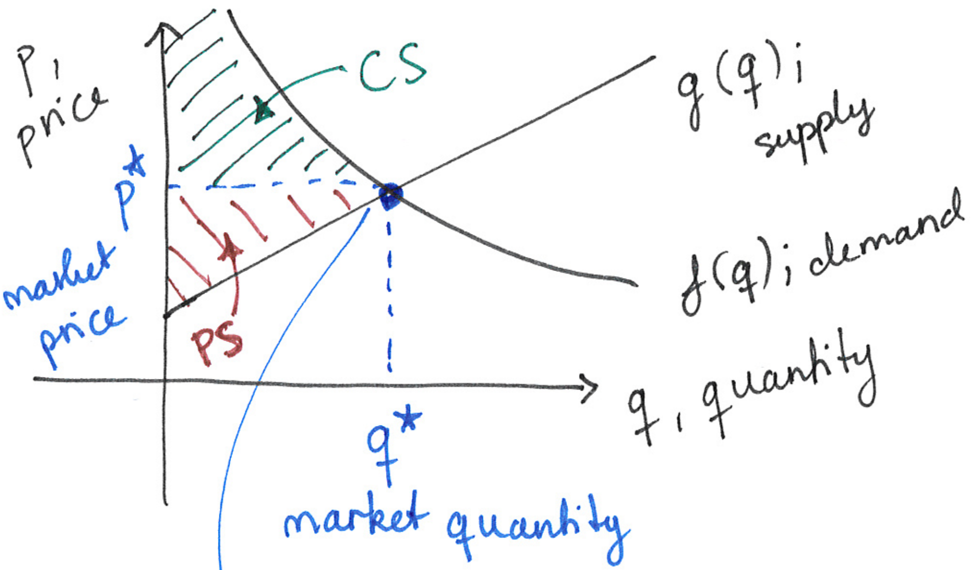
$f(x)$: cash flow per time unit

NPV of cash flow: $\int_0^T f(x)\,e^{-rx}\,dx$

r: discount rate

Consumer / producer surplus

CS / PS



- $g(q)$, supply function (inverse)
- $f(q)$, demand function (inverse)

$$g(q^*) = f(q^*)$$

CS: Consumer surplus:

$$CS = \int_0^{q^*} f(q) - p^*\,dq$$

PS: Producer surplus

$$PS = \int_0^{q^*} p^* - g(q)\,dq$$