INTEGRATION

Partial fractions

EBA1180
Lect. 4
(28)
S25

Ex: $\int \frac{2}{1-x^2}\,dx$

$$\frac{2}{\underbrace{1-x^2}_{(1+x)(1-x)}} = \frac{A}{1+x} + \frac{B}{1-x}$$

$$0\cdot x + 2 = (B-A)x + (A+B)$$

1) $0 = B - A$
2) $2 = A + B$

} 2 linear eqns & 2 unknowns $\Rightarrow A = B = 1$

Q: $\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x} + \frac{1}{1-x}\,dx$

Partial fractions: Above + last time

$$= \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx$$

$$= \frac{1}{1}\ln|1+x| + \frac{1}{(-1)}\ln|1-x| + C$$

SUBSTITUTION:
$u = 1-x$
$du = -dx$
$dx = \frac{1}{-1}du$

$$= \ln|1+x| - \ln|1-x| + C$$

$$= \ln\frac{|1+x|}{|1-x|} + C$$

Problem set 27

1) MET / EBA 1180 Spring 17 :

$$f(x) = 0{,}6 \ln(1+x) + 0{,}4 \ln(1-x), \quad \boxed{0 \le x < 1}$$

a) $$f'(x) = 0{,}6 \frac{1}{1+x} \cdot \underbrace{1}_{\text{from chain rule}} + 0{,}4 \frac{1}{1-x} \cdot \underbrace{(-1)}_{\text{from chain rule}}$$

NB: No issues with division by 0 because of domain of def / bounds on x

$$= \frac{0{,}6}{1+x} - \frac{0{,}4}{1-x}$$

Common denominator

$$= \frac{0{,}6(1-x) - 0{,}4(1+x)}{(1+x)(1-x)}$$

$$= \frac{0{,}6 - 0{,}6x - 0{,}4 - 0{,}4x}{(1+x)(1-x)}$$

$$= \frac{0{,}2 - x}{(1+x)(1-x)}$$

Will be interested in sign of f': Want a factorized form

So: $f'(x) = 0$ gives:

$$\frac{0,2 - x}{(1+x)(1-x)} = 0$$

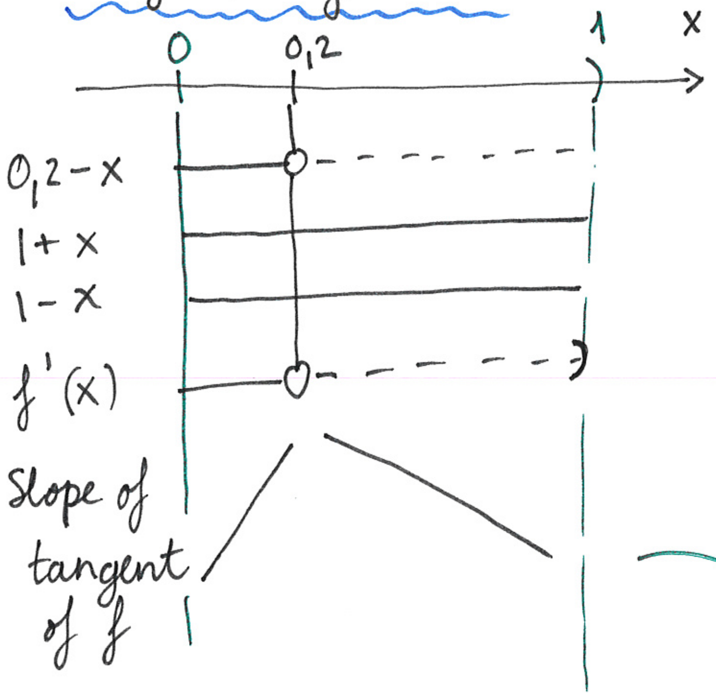
$$0,2 - x = 0$$

$$x = 0,2$$

Candidate for the max. point: Need to check whether it actually is max. point.

To do so: Find sign of derivative

Sign diagram:



$$f'(x) = \frac{0,2 - x}{(1+x)(1-x)}$$

Cap diagram between 0 and 1 because of domain

From this, our candidate point $x = 0,2$ is in fact a (global) max. point, so

$$x^* = 0,2$$

The max. value:

$$f(x^*) = 0,6 \ln(1,2) + 0,4 \ln(0,8)$$

Calculator

$$\approx 0,0201$$

approximately equal

b) $$f''(x) = \left(\frac{0{,}2 - x}{1+x} \cdot \frac{1}{1-x} \right)'$$

$$= \underbrace{\frac{(-1)(1+x) - (0{,}2-x)\cdot 1}{(1+x)^2} \cdot \frac{1}{1-x}}$$

Quotient rule combined with product rule

$$+ \frac{0{,}2-x}{1+x} \, \frac{1}{(1-x)^2} \underbrace{(-1)^2}_{\text{product rule + chain rule}}$$

START 13.00

$$= \frac{(-1 - \cancel{x} - 0{,}2 + \cancel{x})(1-x) + (0{,}2-x)(1+x)}{(1+x)^2(1-x)^2}$$

$$= \frac{-1{,}2 + 1{,}2x + 0{,}2 + 0{,}2x - x - x^2}{(1+x)^2(1-x)^2}$$

$$= \frac{-1 + 0{,}4x - x^2}{(1+x)^2(1-x)^2}$$

Sign of numerator: $-x^2 + 0{,}4x - 1 = 0 \quad | \cdot (-5)$

$$5x^2 - 2x + 5 = 0$$

abc-formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{2 \pm \sqrt{4 - 4\cdot 5 \cdot 5}}{2 \cdot 5}$$

$$= \frac{2 \pm \sqrt{4 - 100}}{10}$$

Negative! No real solutions

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So: $-x^2 + 0{,}4x - 1$ is never 0.

Is it positive or negative?

$$x = 0 \Rightarrow -0^2 + 0{,}4 \cdot 0 - 1 = -1 < 0$$

Negative!

$\boxed{\text{Numerator is negative.}}$

Denominator sign:

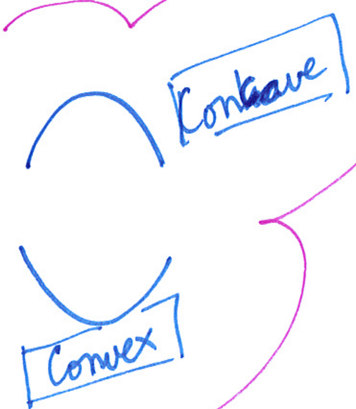
Also $(1-x)^2 > 0$ and $(1+x)^2 > 0$, hence

Because of domain: $0 \le x < 1$

$\boxed{\text{denominator is positive}}$ $(+ \cdot + = +)$

$\frac{-}{+} = -$

Hence; $$f''(x) < 0 \text{ for all } x.$$



Hence, f is concave.

0,4

In a): 0,2

c) Show $f(x) < 0$ for $x > 2 \cdot x^*$:

From a), $f'(x) < 0$ for $x > x^* = 0{,}2$.

See sign diagram

Hence, f is decreasing for $x > x^* = 0{,}2$.

Also,

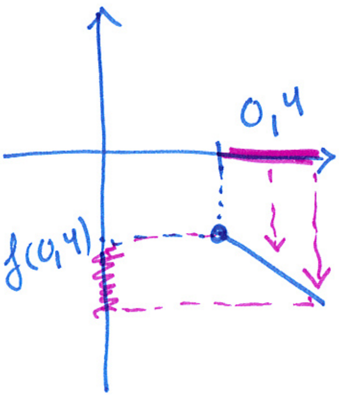
$$f(2x^*) = f(0{,}4) = 0{,}6 \ln(1{,}4) + 0{,}4 \ln(0{,}6)$$

Calculator

$$\approx -0{,}0024 < 0$$

So: $f(\underbrace{2x^*}_{0,4}) < 0$ and $f'(x)$ decreases for all $x > \underbrace{x^*}_{0,2}$. In particular for $x > \underbrace{2x^*}_{0,4}$

Therefore, $f(x) < 0$ when $x > \underbrace{2x^*}_{0,4}$.



d) Sketch graph: We know:

- $f(\underbrace{0,2}_{x^*}) \approx \underbrace{0,0201}_{\text{max. value}}$ → From a)

- $f(\underbrace{0,4}_{2x^*}) \approx -0,0024$ → From c)

- f is increasing for $x < 0,2$ and decreasing for $x > 0,2$. → From sign diagram in a)

- f is concave. → From b)

- f is defined on $[0, 1)$ → $[0, 1\rangle$ → From exercise text

Where does f start?

$$f(0) = 0,6 \ln(1+0) + 0,4 \ln(1-0)$$
$$= 0,6 \underbrace{\ln(1)}_{0} + 0,4 \underbrace{\ln(1)}_{0} = \underline{0}$$

What happens when we approach 1?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \underbrace{0{,}6 \ln(\underbrace{1+x}_{\to 2})}_{\to 0{,}6\ln(2)} + \underbrace{0{,}4 \ln(\underbrace{1-x}_{\to 0})}_{\to -\infty}$$

From below since $D_f = [0,1)$

DRAW ALL OF THIS!

$$= -\infty$$

" $0{,}6 \ln 2 + (-\infty) = -\infty$ "