

- Plan
1. Implicit differentiation
 2. The second derivative and curvature
 3. Convex optimization

1. Implicit differentiation

EX A curve is implicitly defined by the equation $y^2 - x^3 = 1$

- a) Express y' by x and y using implicit differentiation.
- b) Find all solutions for y when $x = 2$.
- c) Compute y' for these points
- d) Find the function expression(s) for the tangent line(s) at $x = 2$.

solution

a) $(y^2)'_x - (x^3)'_x = (1)'_x$

{ chain rule: $u = u(x) = y$ and $g(u) = u^2$
 $u'(x) = y'_x$ $g'(u) = 2u$

{ so $(y^2)'_x = 2u \cdot u'_x = 2y \cdot y'$

$2y \cdot y' - 3x^2 = 0$ - solve for y'
 $2y \cdot y' = 3x^2 \quad | : 2y$

$$y' = \frac{3x^2}{2y}$$

b) $x=2$, solve $y^2 - 2^3 = 1$

$$y^2 = 1 + 8 = 9$$

$$\underline{\underline{y = \pm 3}}$$

c) $(2, 3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot 3} = \underline{\underline{2}}$

$(2, -3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot (-3)} = \underline{\underline{-2}}$

d) Tangent line through $(2, 3)$:

$$h(x) - 3 = 2 \cdot (x - 2)$$

$$\text{so } \underline{\underline{h(x) = 2x - 1}}$$

Tangent line through $(2, -3)$:

$$g(x) - (-3) = -2 \cdot (x - 2)$$

$$\text{so } \underline{\underline{g(x) = -2x + 1}}$$

2. The second order derivative and curvature

bending up

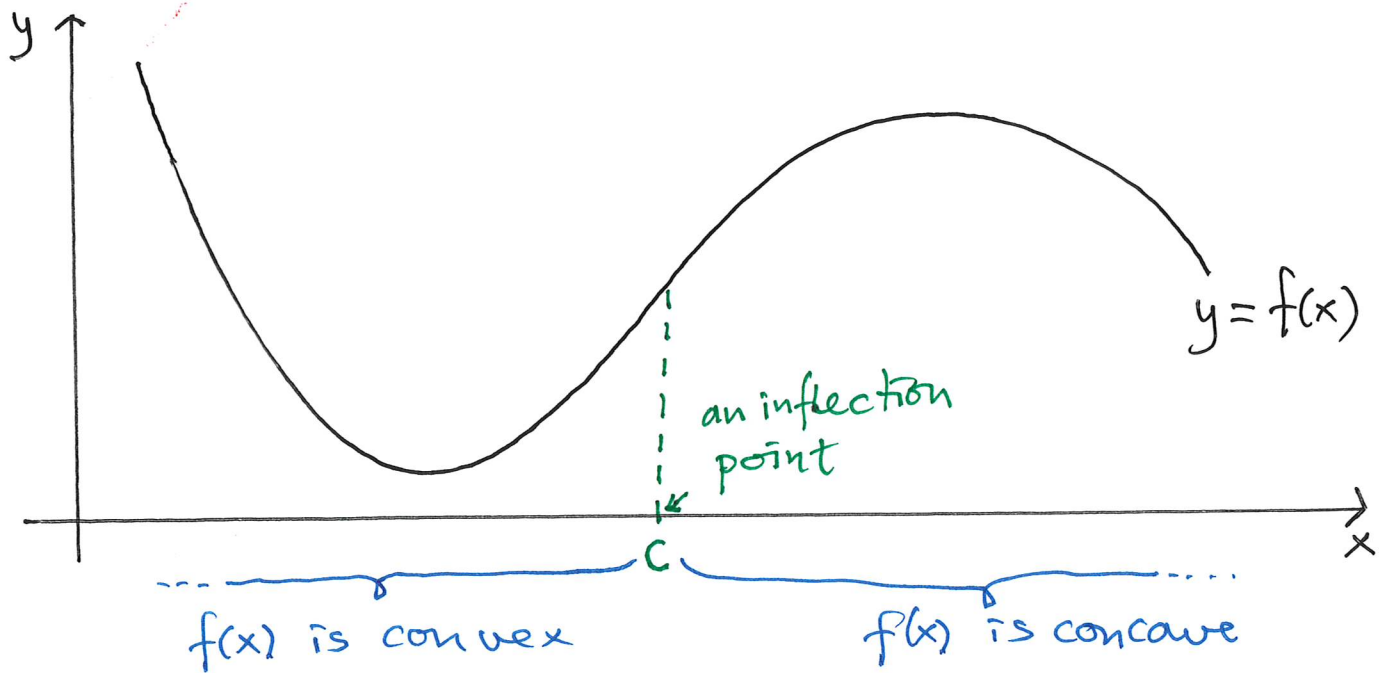


3 graphs of convex functions

bending down



3 graphs of concave functions



Definition

- $f(x)$ is convex in the interval $[a, b]$ if $f''(x) \geq 0$ for all x in (a, b) .
- $f(x)$ is concave in the interval $[a, b]$ if $f''(x) \leq 0$ for all x in (a, b) .
- A number c is an inflection point for $f(x)$ if $f''(x)$ changes sign at $x = c$.

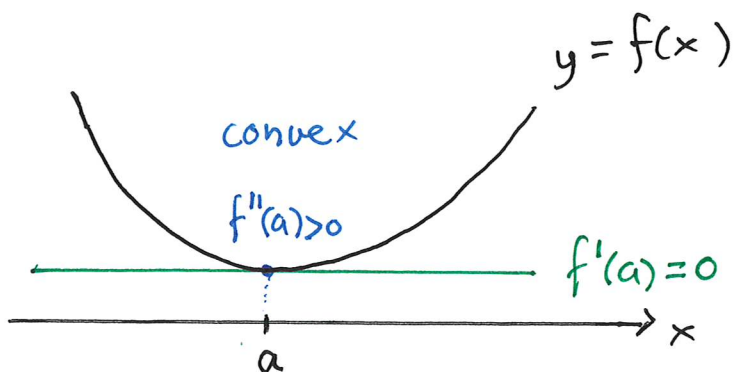
Note If $f(x)$ is convex, then $f'(x)$ is an increasing function.

If $f(x)$ is concave, then $f'(x)$ is a decreasing function.

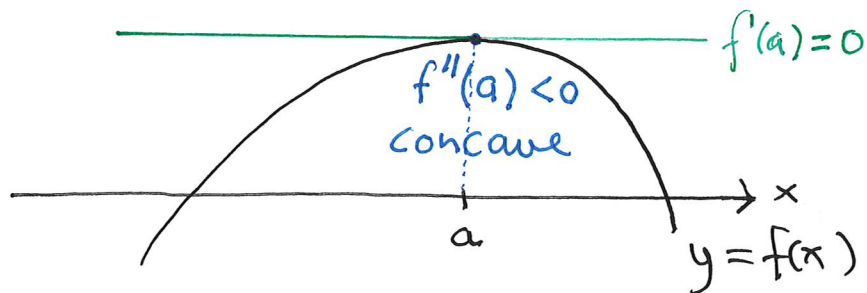
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second derivative test (Sec. 8.5)

Suppose $x=a$ is a stationary point for $f(x)$.
If $f''(a) > 0$ then $x=a$ is a (local) minimum point.



If $f''(a) < 0$ then $x=a$ is a (local) maximum point.



Ex $f(x) = x^3 - 3x^2 + 5$. Find local max/min. points by using the second derivative test.

Solution Calculate $f'(x) = 3x^2 - 6x$.

Stationary point: Solutions of the eq. $f'(x) = 0$

e.g. $3x^2 - 6x = 0$ ($3x$ is a common factor)

$$3x(x - 2) = 0$$

either $x=0$ or $x=2$

$$\begin{aligned}\text{Calculate } f''(x) &= [f'(x)]' = (3x^2 - 6x)' \\ &= \underline{6x - 6}\end{aligned}$$

$$f''(0) = 6 \cdot 0 - 6 = -6 < 0$$

so $x = 0$ is a (loc.) maximum point.

$$f''(2) = 6 \cdot 2 - 6 = 6 > 0$$

so $x = 2$ is a (loc.) minimum point.

3. Convex optimization

Facts • If $f(x)$ is convex everywhere in its domain (an interval), then any stationary point is a global minimum point.

• If $f(x)$ is concave everywhere in its domain (an interval), then any stationary point is a global maximum point.

Ex $f(x) = x^4 + 5x^2 + 3$, $D_f = \langle \leftarrow, \rightarrow \rangle = \langle \infty, \infty \rangle = \mathbb{R}$

a) Find the stationary points.

b) Determine if they are global max. or min. points.

c) Determine the extremal values.

Solution

a) Calculate $f'(x) = 4x^3 + 10x$

Stationary points are solutions to the eq.

$$4x^3 + 10x = 0 \quad (x \text{ is a common factor})$$

$$x(4x^2 + 10) = 0$$

so $x = 0$ is the only stationary point.

b) Calculate $f''(x) = (4x^3 + 10x)' = 12x^2 + 10$

which is greater or equal to 10 for all x.

So $x = 0$ is a global min. point for $f(x)$. (The graph of $f(x)$ is convex.)

c) $f(0) = 0^4 + 5 \cdot 0^2 + 3 = \underline{\underline{3}}$ is the global minimum value. There are no global max. values.