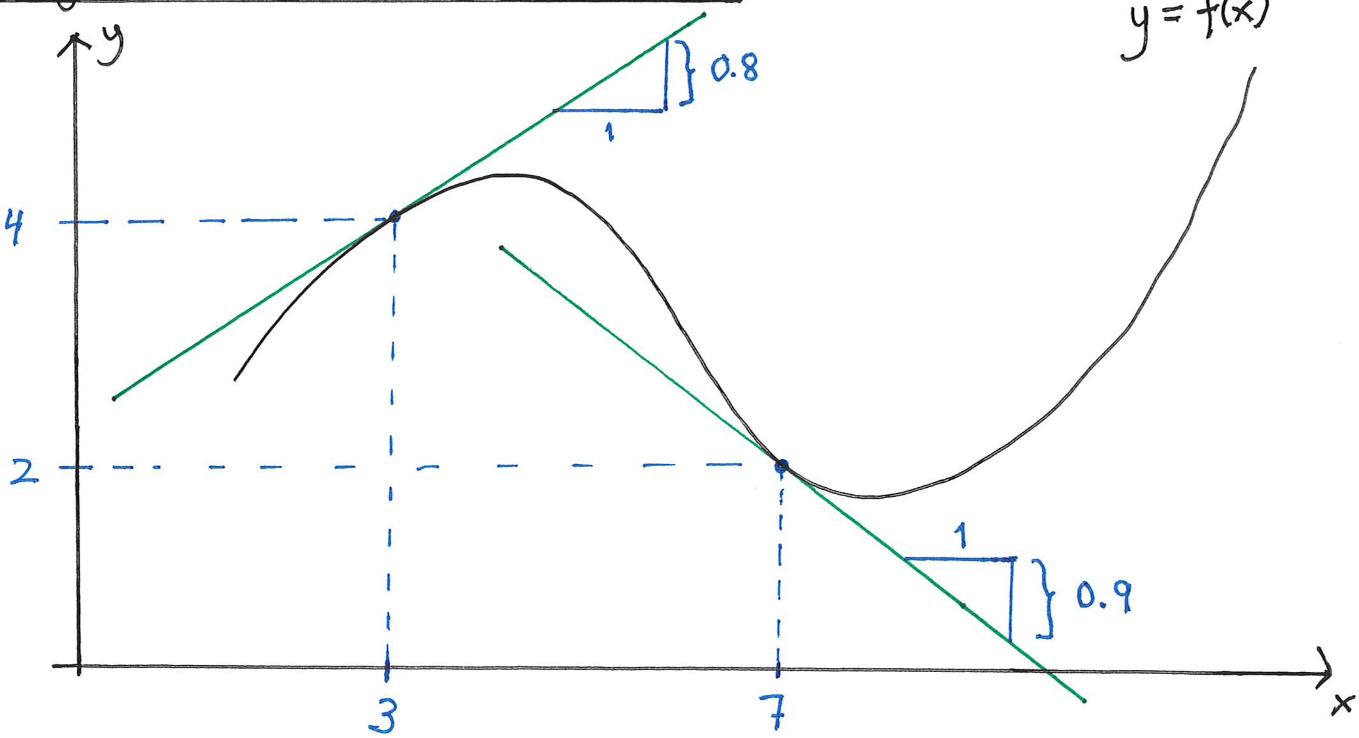
EBA 1180, lecture 16, 16 Oct. 2024, Rumar Ile

Plan
1. Tangents and the derivative
2. The derivative as a function
3. Rules for differentiation

1. Tangents and the derivative

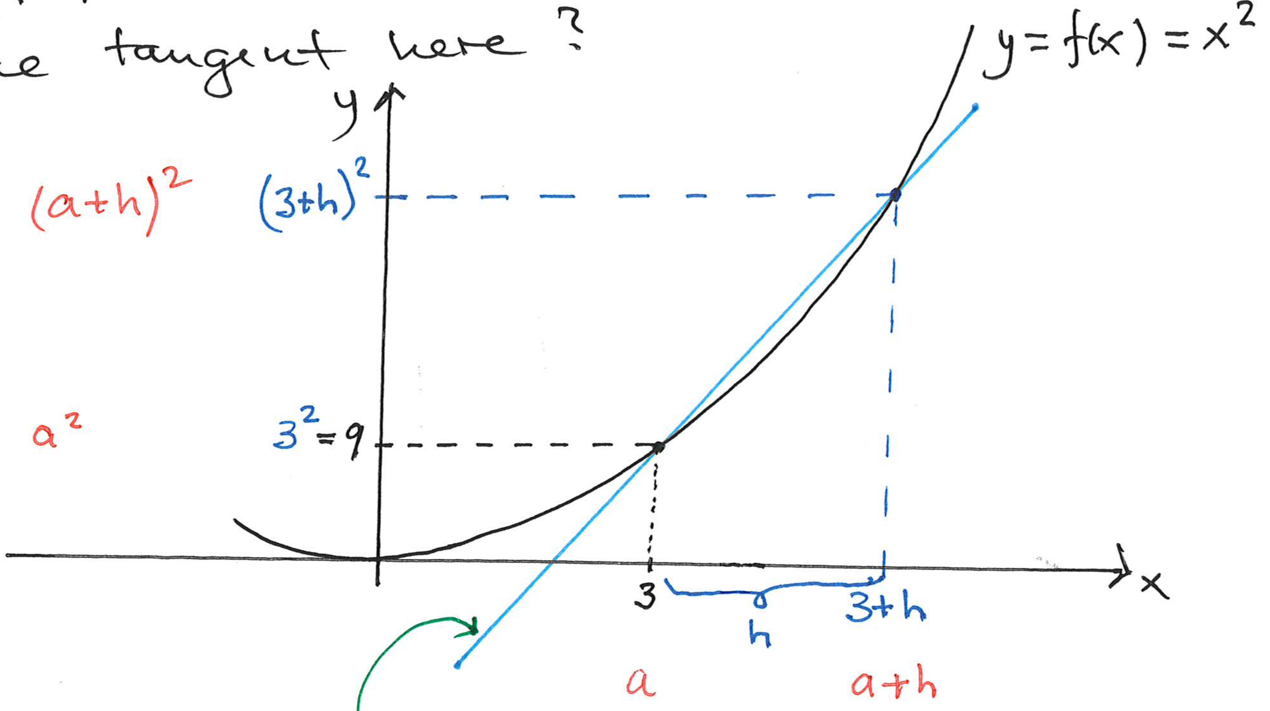


- The tangent of the graph of $f(x)$ at the point $(3,4)$ has slope 0.8.
  We write $f'(3) = 0.8$
- The tangent of the graph of $f(x)$ at the point $(7, 2)$ has slope -0.9.
  We write $f'(7) = -0.9$

Two important applications
1) To determine where the function $f(x)$ increases/decreases and where max/min is.
2) Approximate complicated functions with linear functions
   – typical for economic models.

How to find the slope of the tangent?

Ex $f(x) = x^2$ and $(3, 9)$. What is the slope of the tangent here?



The slope of this secant line is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{(3+h)^2 - 3^2}{h} = \frac{(3+h)(3+h) - 9}{h}$$

$(a+h)^2 - a^2$ $\quad (a+h)(a+h) - a^2$

$a^2 + 2ah + h^2 - a^2 \quad h^2 + 2ah \quad h(h+2a)$

$$= \frac{9 + 2\cdot 3h + h^2 - 9}{h} = \frac{h^2 + 6h}{h} = \frac{h(h+6)}{h}$$

$h + 2a \xrightarrow[h \to 0]{} 2a$

$= h + 6 \xrightarrow[h \to 0]{} 6$ which has to be the slope of the tangent line to $f(x)$ throgh $(3, 9)$.

We write $f'(3) = 6$

also $f'(a) = 2a$

2. The derivative as a function

In the example: If $x = a$ then $f'(a) = 2a$
– this is a function. We use x as variable:

$$f'(x) = 2x$$

E.g. the slope of the tangent of $f(x)$

- at $(-3, 9)$ is $f'(-3) = 2\cdot(-3) = -6$
- at $(1, 1)$ is $f'(1) = 2\cdot 1 = 2$
- at $(10, 100)$ is $f'(10) = 2\cdot 10 = 20$

We could do a similar calculation with $f(x) = x^3$ and get $f'(x) = 3x^2$

---

3. Rules of differentiation

Start: 11.05

<u>Power rule</u> $\quad f(x) = x^n$ gives $f'(x) = n\cdot x^{n-1}$

Note: this is true for <u>all</u> n.

<u>Ex</u> $f(x) = x^{10}$, $\quad f'(x) = 10\cdot x^9 \quad (n = 10)$

<u>Ex</u> $f(x) = \sqrt[3]{x}$, $\quad f'(x) = \frac{1}{3}\cdot x^{\frac{1}{3}-1} \quad (n = \frac{1}{3})$

$= x^{\frac{1}{3}}$

$$= \frac{1}{3}\cdot x^{-\frac{2}{3}}$$

$$= \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3}\cdot\frac{1}{\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{x^2}}$$

The sum rule If $f(x) = g(x) + h(x)$
then $f'(x) = g'(x) + h'(x)$

Ex $f(x) = x + x^3$ then $f'(x) = 1 + 3x^2$

The constant rule If k is a constant number and $f(x) = k \cdot g(x)$ then

$$f'(x) = k \cdot g'(x)$$

Ex $k = 7$, $g(x) = x^2$, then $f(x) = 7x^2$
and $f'(x) = 7 \cdot 2x = 14x$

The product rule

If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Ex $f(x) = (5x^3 - 2x + 1) \cdot (3x + 7)$

Calculate the $f'(x)$ by using the product rule.

Solution $g(x) = 5x^3 - 2x + 1$ and $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $\qquad$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2) \cdot (3x + 7) + (5x^3 - 2x + 1) \cdot 3$

note the parentheses !

calculate

$$= \underline{\underline{60x^3 + 105x^2 - 12x - 11}}$$

The quotient rule

Suppose $f(x) = \dfrac{g(x)}{h(x)}$

Then $$f'(x) = \frac{g'(x)\cdot h(x) - g(x)\cdot h'(x)}{[h(x)]^2}$$

Ex $f(x) = \dfrac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$

$g'(x) = 3$ $\qquad$ $h'(x) = 2$

$$f'(x) = \frac{3\cdot(2x+5) - (3x+1)\cdot 2}{(2x+5)^2}$$

note the parentheses!

note this sign! - for the whole parenthesis

$$= \frac{6x+15-6x-2}{(2x+5)^2}$$

$$= \frac{13}{(2x+5)^2}$$

← usually better to keep the parenthesis in the denominator

The chain rule

the inner function

If $f(x) = g(u(x))$

the outer function

then $f'(x) = g'(u) \cdot u'(x)$ where $u = u(x)$

Ex $f(x) = (x^2+3)^{10}$

Put $u = u(x) = x^2+3$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\qquad g'(u) = 10u^9$

Then $f'(x) = 10u^9 \cdot 2x = 10 \cdot (x^2+3)^9 \cdot 2x$

$= \underline{\underline{20x(x^2+3)^9}}$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\qquad g'(x) = \frac{1}{x}$

Ex $f(x) = e^{3x}$

Chain rule: $u = 3x$ and $g(u) = e^u$

$u'(x) = 3$ $\qquad g'(u) = e^u$

$f'(x) = e^u \cdot 3 = \underline{\underline{3e^{3x}}}$

Ex $f(x) = \ln(x^2+1)$

Chain rule: $u = x^2+1$, $g(u) = \ln(u)$

$u'(x) = 2x$ $\qquad g'(u) = \frac{1}{u}$

$f'(x) = \frac{1}{u} \cdot 2x = \underline{\underline{\frac{2x}{x^2+1}}}$