EBA 1180, lecture 15, 15 Oct. 2024, Runar Ile

Plan: Repetition

1. Inverse functions
2. Logarithmic- and exponential functions
3. Asymptotes

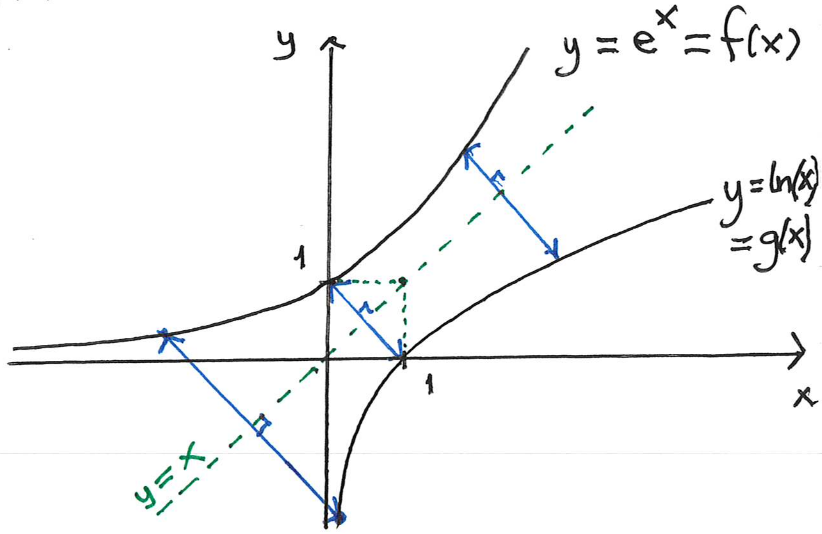
1. Inverse functions

Definition:

$f(g(x)) = x$ for all x in D_g

$g(f(x)) = x$ for all x in D_f

*) The graphs are symmetric around the line $y=x$



*) For $f(x)$ to have an inverse function $f(x)$ has to be strictly increasing or strictly decreasing.

*) $D_g = R_f$ and $R_g = D_f$

How to find $g(x)$ and D_g in practice.

Probl 5d $\quad f(x) = 20 + \frac{1}{x-3}$, $D_f = \langle 3, \rightarrow \rangle$

We find $g(x)$ and D_g.

1 Solve the eq $y = f(x)$ for x.

that is $\quad y = 20 + \frac{1}{x-3} \quad | \cdot (x-3)$

$$y \cdot (x-3) = 20 \cdot (x-3) + 1$$

$$yx - 3y = 20x - 60 + 1 = 20x - 59$$

$$yx - 20x = 3y - 59$$

$$(y-20)x = 3y - 59 \quad | : (y-20)$$

$$x = \frac{3y-59}{y-20}$$

$$x \overset{\text{poly. div.}}{=} 3 + \frac{1}{y-20}$$

(2) Exchanges variables $(y \leftrightarrow x)$

$$y = \underline{\underline{g(x) = 3 + \frac{1}{x-20}}}$$

(3) Put $D_g = R_f$ and find R_f.

R_f = the set of function values of $f(x)$ when $x \in D_f$

Note that $f(x) \xrightarrow[x \to 3^+]{} +\infty$ and

$f(x) \xrightarrow[x \to \infty]{} 20^+$ so $D_g = R_f = \underline{\underline{\langle 20, \rightarrow \rangle}}$

(alt: $20 < x$

alt.: $x > 20$)

2. Logarithmic and exponential functions.

Probl. 6 Given $\ln(2) = 0.6931$, $\ln(3) = 1.0986$

and $\ln(5) = 1.6094$. Then (without ln on the the calc.)

d) $$\ln \frac{1\,000\,000}{27} = \ln(10^6) - \ln(3^3)$$

$$= 6 \cdot \ln(10) - 3 \cdot \ln(3)$$

$$= 6(\ln(2) + \ln(5)) - 3 \cdot \ln(3)$$

$$= 6 \cdot (0.6931 + 1.6094) - 3 \cdot 1.0986$$

$$= \underline{\underline{10.5192}}$$

f) $$\ln\left(\sqrt[10]{6}\right) = \ln\left(6^{\frac{1}{10}}\right) = \frac{1}{10} \cdot \ln(6)$$

$$= \frac{\ln(2) + \ln(3)}{10} = \underline{\underline{0.1792}}$$

$f(x) = a^x$, D_f = all numbers on the number line,

$a > 0$, $a \neq 1$

$g(x) = \log_a(x)$, $D_g = \langle 0, \rightarrow \rangle = R_f$

Ex How long time will it take to double the deposit on an account with 3 % interest?

Solution: $f(x) = 1.03^x$ is the balance after x years if the deposit was 1.
We have to solve the eq.

$$1.03^x = 2 \qquad (*)$$

then $$x = \underline{\underline{\log_{1.03}(2)}}$$

But, we cannot put this into the calculator directly.
Instead we put the LHS and RHS of (*) into $\ln(x) = \log_e(x)$.

get $$\ln(1.03^x) = \ln(2)$$

$$x \cdot \ln(1.03) = \ln(2) \qquad | : \ln(1.03)$$

$$x = \frac{\ln(2)}{\ln(1.03)} \approx \underline{\underline{23.45}}$$

This also means that

$$\log_{1.03}(2) = \frac{\ln(2)}{\ln(1.03)}$$

Pattern $$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Start: 11.05

Probl. 8c $\quad \ln(x-3) < -2$

Since e^x is strictly increasing we can put the LHS and RHS into e^x and get an equivalent inequality

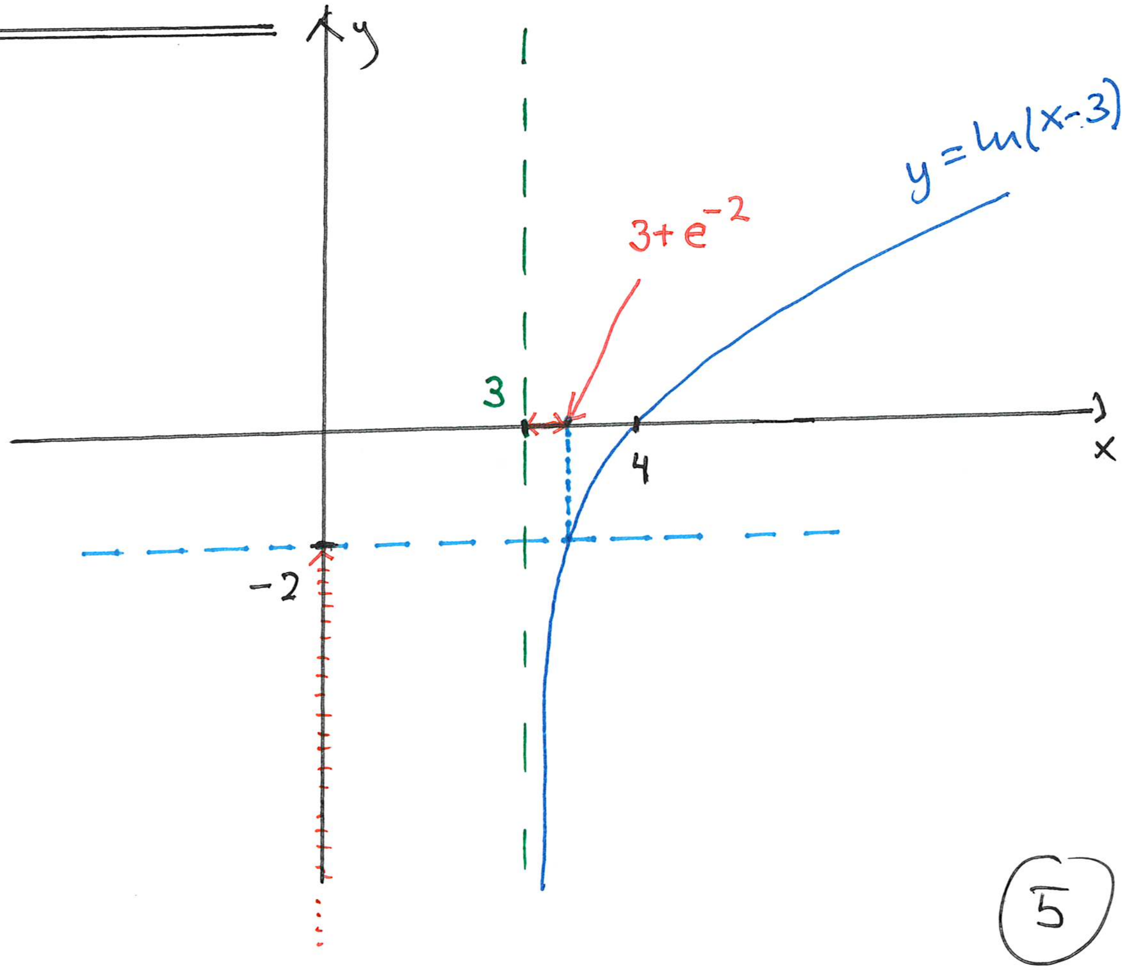
$$e^{\ln(x-3)} < e^{-2}$$

$$x-3 < e^{-2}$$

$$x < 3+e^{-2}$$

But note that the original inequality is only defined for $x>3$. So the set of solutions is

$$\langle 3, 3+e^{-2} \rangle$$



Probl 8e $\dfrac{3e^x}{e^x+1} < 5$

We could solve this by putting $u = e^x$ to get

$$\frac{3u}{u+1} < 5$$

$$\frac{3u}{u+1} - 5 < 0$$

- one fraction
- sign. diag.
- use $u = e^x$

But here it is simpler to multiply each side with $e^x + 1$. Because $e^x + 1$ is greater than 0 for all x, this gives an equivalent inequality

$$3 \cdot e^x < 5(e^x+1) = 5e^x + 5$$

$$-5 < 2e^x \quad | : 2$$

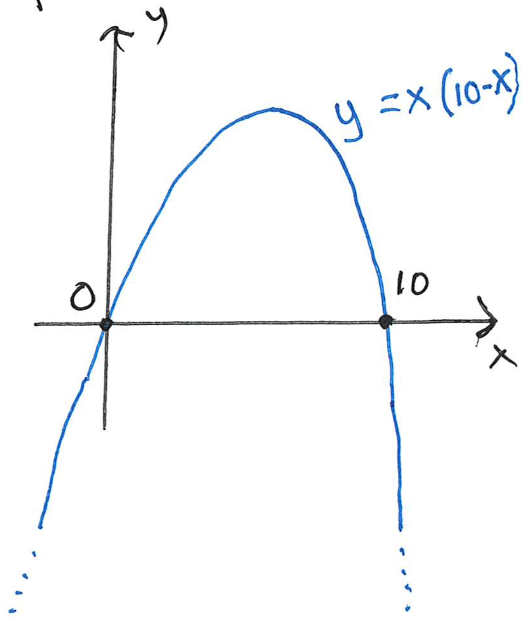
$$-\frac{5}{2} < e^x$$

and this is true for all values of x.

3. Asymptots

Probl 9 Determine the asymptots of $f(x)$.

b) $f(x) = e^{x(10-x)} + 50$



We note that $x(10-x) \xrightarrow[x \to \pm\infty]{} -\infty$

Hence $e^{x(10-x)} \xrightarrow[x \to \pm\infty]{} 0^+$

and so $f(x) \xrightarrow[x \to \pm\infty]{} 50^+$

so $y = 50$ is a horizontal asymptote for $f(x)$.

9f) $f(x) = \ln(\overbrace{120x+10}^{\text{pos. in } D_f}) - \ln(\overbrace{20x-30}^{\text{pos. in } D_f})$

with $D_f = \langle \frac{3}{2}, \rightarrow \rangle$

$= \ln\left(\frac{120x+10}{20x-30}\right)$

Note that $\frac{120x+10}{20x-30} \xrightarrow[x\to\infty]{} \frac{120}{20} = 6$

so $f(x) \xrightarrow[x\to\infty]{} \ln(6)$ and $\underline{\underline{y = \ln(6)}}$ is a horizontal asymptote.

Note also $\frac{120x+10}{20x-30} \xrightarrow[x\to\frac{3}{2}^+]{} \infty$

Then $f(x) \xrightarrow[x\to\frac{3}{2}^+]{} \infty$ so $\underline{\underline{x = \frac{3}{2}}}$ is a vertical asymptote.

<u>Probl 10</u> Find the inverse function $g(x)$ and D_g.

c) $f(x) = e^{\frac{2}{x+10}}$, $D_f = [0, \rightarrow\rangle$

(1) Solve the eq. $e^{\frac{2}{x+10}} = y$ for x.

$$\frac{2}{x+10} = \ln\left(e^{\frac{2}{x+10}}\right) = \ln(y) \quad | \cdot (x+10)$$

$$2 = \ln(y)\cdot(x+10) = \ln(y)\cdot x + 10\cdot\ln(y)$$

$$2 - 10\ln(y) = \ln(y)\cdot x \quad \Big|\ :\ln(y)$$

$$x = \frac{2-10\ln(y)}{\ln(y)} = \frac{2}{\ln(y)} - 10$$

(2) $$g(x) = \frac{2}{\ln(x)} - 10$$

(3) $D_g = R_f$. Note $\frac{2}{x+10}$ is a decreasing function. So max. is

$$f(0) = e^{\frac{2}{0+10}} = e^{\frac{1}{5}}$$

and $f(x) \xrightarrow[x\to\infty]{} e^{0^+} = 1$

so $D_g = R_f = \left\langle 1, e^{\frac{1}{5}}\right]$