

*I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.*

R. Lucas

## Lecture 19 – 20

Sec. 7.1, 6.9, 8.6-7:

**Implicit differentiation. The second order derivative, convex/concave functions.**

Here are recommended exercises from the textbook [SHSC].

Section 7.1 exercise 1, 4, 6, 7a

Section 6.9 exercise 1-4

Section 9.6 exercise 1-4, 6a

Section 8.6 exercise 1-4

### Problems for the exercise session Wednesday 1 Nov. 12–17+ in B1-065

**Problem 1** Find an expression for  $y'$  in terms of  $y$  and  $x$  by implicit differentiation. Find all solutions for  $y$  with  $x = a$  and determine the expression for the tangent function in each of these points.

a)  $x^2 + 25y^2 - 50y = 0$  and  $a = 4$

b)  $x^{3.27}y^{1.09} = 1$  and  $a = 1$

c)  $x^4 - x^2 + y^4 = 0$  and  $a = \frac{\sqrt{2}}{2}$

d)  $x^3 - 3xy + y^2 = 0$  and  $a = 2$

**Problem 2** in figure 1 you see the graphs of the implicitly defined curves in Problem 1. Determine the curves and the equations which belong together. Also draw the tangents in Problem 1.

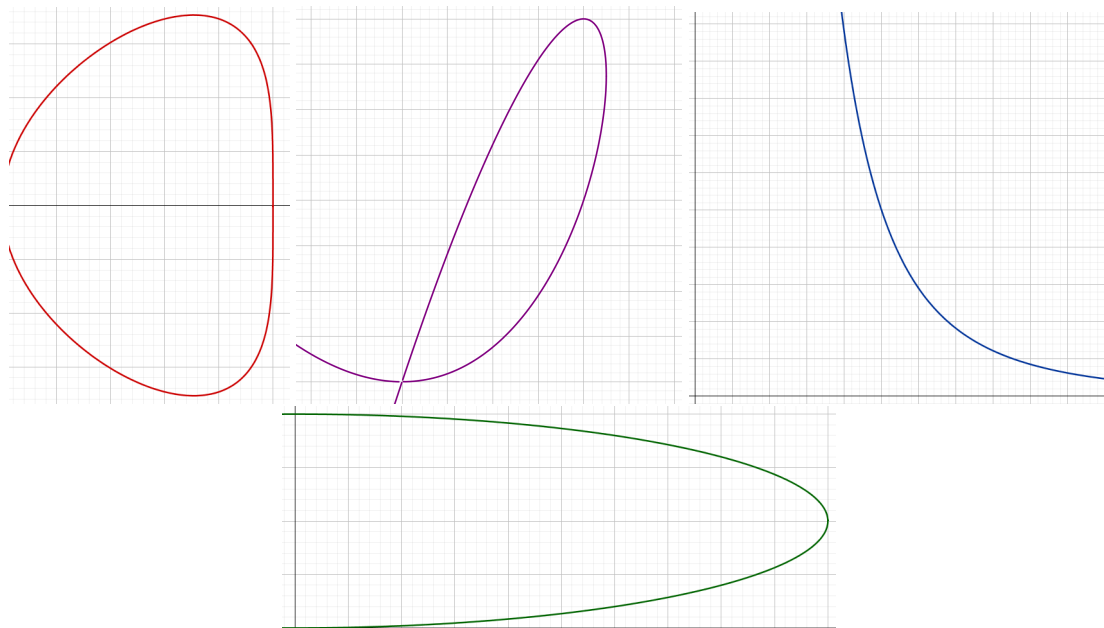


Figure 1: Four implicitly defined curves

**Problem 3** Make a sketch of the graphs of **TWO** different functions  $f(x)$  with the given data. One of the functions should be *strictly increasing*. Note: You are not supposed to find any algebraic expression!

- a)  $f''(x)$  is negative for  $x < 5$  and positive for  $x > 5$
- b)  $f''(x)$  is positive for  $x < 10$ , negative for  $10 < x < 15$  and positive for  $x > 15$

**Problem 4** in figure 2 you see the graph of  $f''(x)$ . Determine if the statement is true or false.

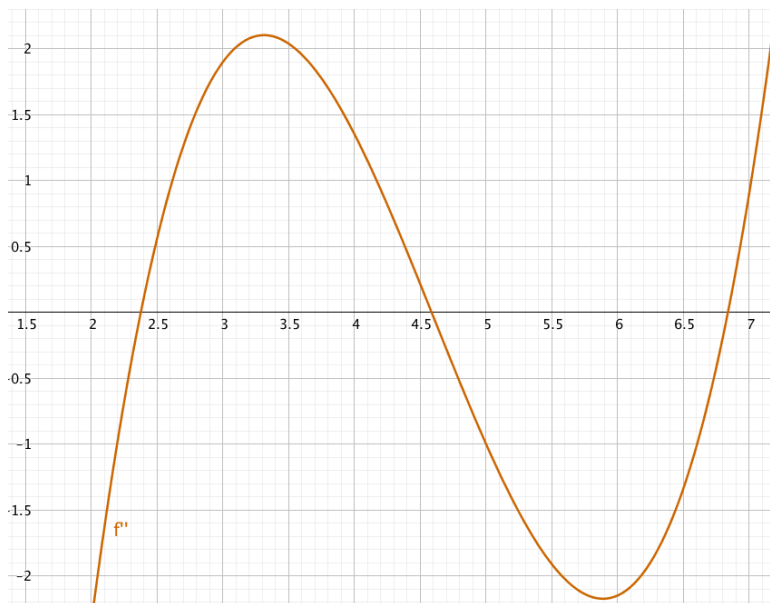


Figure 2: The graph of  $f''(x)$

- a)  $f''(2.5) > f''(4)$
- b)  $f(x)$  is convex for  $3 \leq x \leq 4$
- c)  $f(x)$  has no inflection points between 5.5 and 6
- d)  $f(x)$  has two inflection points for  $2 \leq x \leq 7$
- e)  $f(x)$  is concave for  $6 \leq x \leq 6.5$
- f)  $f'(4)$  is the maximum of  $f'(x)$  for  $x \in [3, 4]$
- g)  $f'(x)$  decreases in the interval  $[4, 5]$
- h)  $f'(x)$  increases faster around  $x = 2.5$  than around  $x = 3$
- i)  $f(4)$  has to be positive
- j)  $f'(2.5) < f'(4.5)$
- k)  $f(x)$  must have at least one minimum point

**Problem 5** In figure 3 you see the graphs of  $f(x)$ ,  $f'(x)$  and  $f''(x)$  in the same coordinate system. Determine which is the graph of  $f(x)$ , of  $f'(x)$  and of  $f''(x)$  in (a-c).

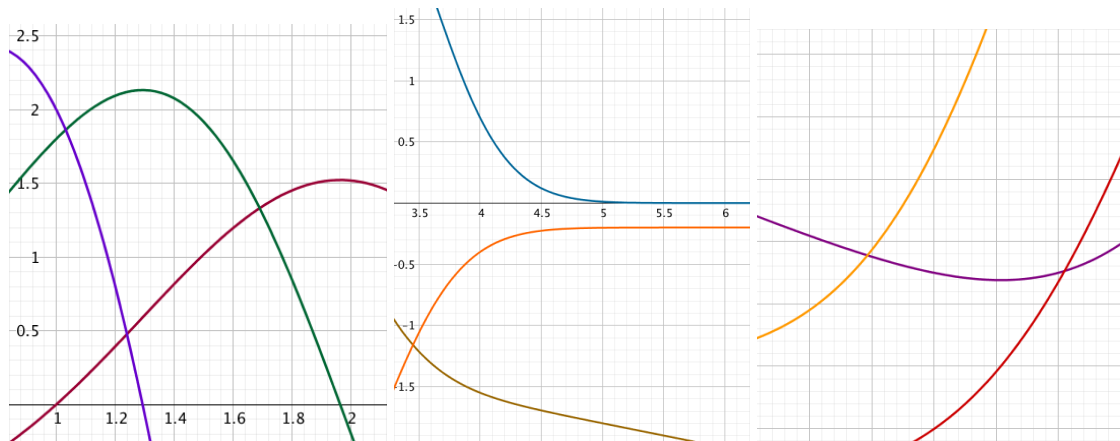


Figure 3: (a-c): The graphs of  $f(x)$ ,  $f'(x)$  and  $f''(x)$

**Problem 6** Calculate  $f'(x)$  and  $f''(x)$ , solve the equation  $f''(x) = 0$ , determine where  $f(x)$  is convex and concave, and determine the inflection points (if any).

a)  $f(x) = x^4 - 8x^3 + 18x^2 + 1$     b)  $f(x) = \ln(x^2 - 2x + 2) - \frac{x}{4} + 1$

c)  $f(x) = e^{-\frac{x^2}{2}} + x + 1$     d)  $f(x) = x^5 - 10x^4 + 30x^3 + 2$

**Problem 7** Determine the expressions for the tangent functions at the inflection points in Problem 6.

**Problem 8** Determine (local) minimum and maximum points for the function  $f(x)$ . Explain why these points give (global) minimum/maximum for  $f(x)$  by using convexity/concavity of the function. Calculate the minimum/maximum of the function.

a)  $f(x) = \ln(-x^2 + 14x - 45)$  with  $D_f = \langle 5, 9 \rangle$     b)  $f(x) = \frac{-1}{x(x-6)}$  with  $D_f = \langle 0, 6 \rangle$     c)  $f(x) = e^{x(x-4)}$  with  $D_f = \mathbb{R}$  (all real numbers)

**Problem 9** (Multiple choice spring 2018, problem 11)

We consider the function  $f(x) = 4\sqrt{x} \ln(x)$ . Which statement is true?

- (A) The function  $f$  has one inflection point
- (B) The function  $f$  has several inflection points
- (C) The function  $f$  is concave
- (D) The function  $f$  is convex
- (E) I choose not to solve this problem.

**Problem 10** Compute the expression for the derivative of  $f(x)$ .

a)  $f(x) = \sqrt{x^2 - 7x + 13}$     b)  $f(x) = xe^{0.1x^2}$   
 c)  $f(x) = (2x + 5)^{100}$     d)  $f(x) = \frac{\ln(x)}{x}$

## Answers

**Problem 1**

- a)  $y' = \frac{-x}{25(y-1)}$ , for  $x = 4$ :  $y = \frac{2}{5}$  or  $y = \frac{8}{5}$  which gives the tangent functions  $h_1(x) = \frac{4}{15}x - \frac{2}{3}$  and  $h_2(x) = -\frac{4}{15}x + \frac{8}{3}$
- b)  $y' = \frac{-3y}{x}$ , for  $x = 1$ :  $y = 1$  which gives the tangent function  $h(x) = -3x + 4$
- c)  $y' = \frac{x(1-2x^2)}{2y^3}$ , for  $x = \frac{\sqrt{2}}{2}$ :  $y = \pm \frac{\sqrt{2}}{2}$  which gives the tangent functions  $h_1(x) = \frac{\sqrt{2}}{2}$  and  $h_2(x) = -\frac{\sqrt{2}}{2}$
- d)  $y' = \frac{3(y-x^2)}{2y-3x}$ , for  $x = 2$ :  $y = 4$  or  $y = 2$  which gives the tangent functions  $h_1(x) = 4$  and  $h_2(x) = 3x - 4$

**Problem 2**

- a) Green    b) Blue    c) Red    d) Purple

**Problem 3**

Compare with other students, ask the learning assistants!

### Problem 4

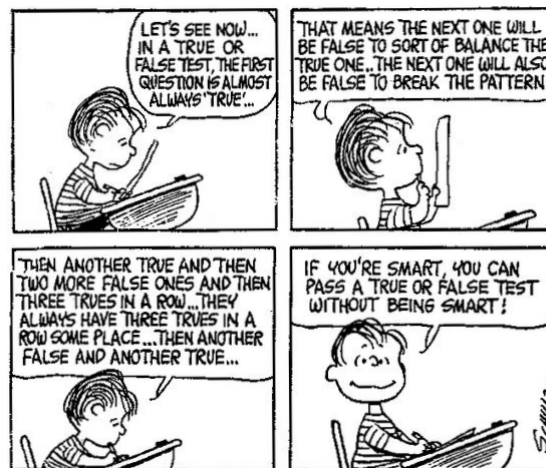


Figure 4: True or false, or opposite

### Problem 5

- a)  $f(x)$ : Dark red,  $f'(x)$ : Green  
 b)  $f(x)$ : Olive,  $f'(x)$ : Orange  
 c)  $f(x)$ : Violet,  $f'(x)$ : Red

### Problem 6

- a)  $f'(x) = 4x^3 - 24x^2 + 36x$  and  $f''(x) = 12(x-1)(x-3)$ .  $f''(x) = 0$  has solutions  $x = 1$  and  $x = 3$ .  $f(x)$  is convex in the interval  $(-\infty, 1]$ ,  $f(x)$  is concave in the interval  $[1, 3]$ , and  $f(x)$  is convex in the interval  $[3, \infty)$ . Hence  $x = 1$  and  $x = 3$  are inflection points.
- b)  $f'(x) = \frac{2x-2}{(x-1)^2+1} - \frac{1}{4}$  and  $f''(x) = \frac{-2x(x-2)}{[(x-1)^2+1]^2}$ .  $f''(x) = 0$  has solutions  $x = 0$  and  $x = 2$ .  $f(x)$  is concave in the interval  $(-\infty, 0]$ ,  $f(x)$  is convex in the interval  $[0, 2]$ , and  $f(x)$  is concave in the interval  $[2, \infty)$ . Hence  $x = 0$  and  $x = 2$  are inflection points.
- c)  $f'(x) = -xe^{-\frac{x^2}{2}} + 1$  and  $f''(x) = (x+1)(x-1)e^{-\frac{x^2}{2}}$ ,  $f''(x) = 0$  has solutions  $x = \pm 1$ ,  $f(x)$  is convex in the interval  $(-\infty, -1]$ ,  $f(x)$  is concave in the interval  $[-1, 1]$ , and  $f(x)$  is convex in the interval  $[1, \infty)$ . Hence  $x = -1$  and  $x = 1$  are inflection points.
- d)  $f'(x) = 5x^4 - 40x^3 + 90x^2$  and  $f''(x) = 20x(x-3)^2$ .  $f''(x) = 0$  has solutions  $x = 0$  and  $x = 3$  (a double root).  $f(x)$  is concave in the interval  $(-\infty, 0]$  and  $f(x)$  is convex in the interval  $[0, \infty)$ . Hence  $x = 0$  is the only inflection point.

### Problem 7

- a) Inflection point tangents:  $h_1(x) = 16x - 4$  and  $h_3(x) = 28$   
 b) Inflection point tangents:  $h_0(x) = -1.25x + \ln(2) + 1$  and  $h_2(x) = 0.75x + \ln(2) - 1$   
 c) Inflection point tangents:  $h_{-1}(x) = (1 + e^{-0.5})x + 2e^{-0.5} + 1$  and  $h_1(x) = (1 - e^{-0.5})x + 2e^{-0.5} + 1$   
 d) Inflection point tangent:  $h_0(x) = 2$

### Problem 8

- a)  $f'(x) = \frac{2(7-x)}{-x^2+14x-45}$  which changes sign from + to - at  $x = 7$ .  $f''(x) = \frac{-2[(x-7)^2+4]}{(-x^2+14x-45)^2}$  is negative for all  $x$ , so  $f(x)$  is concave, max:  $f(7) = 2\ln(2) = 1.39$
- b)  $f'(x) = \frac{2x-6}{x^2(x-6)^2}$  which changes sign from - to + at  $x = 3$ .  $f''(x) = \frac{-6[(x-3)^2+3]}{x^3(x-6)^3}$  is positive for all  $x \in (0, 6)$ , so  $f(x)$  is convex, min:  $f(3) = \frac{1}{9} = 0.11$
- c)  $f'(x) = 2(x-2)e^{x(x-4)}$  which changes sign from - to + at  $x = 2$ .  
 $f''(x) = 4[(x-2)^2 + \frac{1}{2}]e^{x(x-4)}$  is positive for all  $x$ , so  $f(x)$  is convex, min:  $f(2) = e^{-4} = 0.02$

### Problem 9 A

### Problem 10

- a)  $f'(x) = \frac{2x-7}{2\sqrt{x^2-7x+13}}$   
 b)  $f'(x) = \frac{1}{5}(x^2+5)e^{0.1x^2}$   
 c)  $f'(x) = 200(2x+5)^{99}$   
 d)  $f'(x) = \frac{1-\ln(x)}{x^2}$