Interpretation of Lagrange multipliers

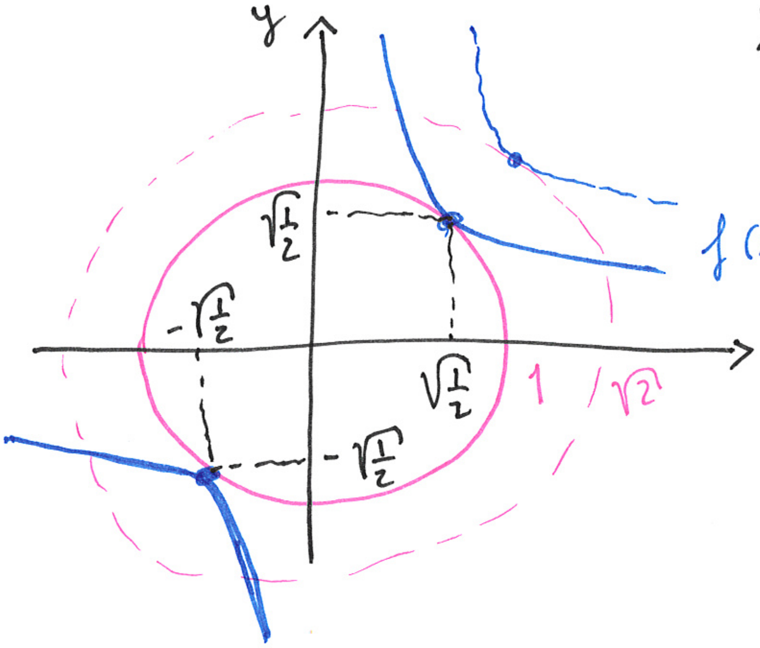
EBA 1180
Sect. 47
Spring 24

<u>Ex</u>: max/min $f(x,y) = xy$ when
$$\underbrace{x^2+y^2}_{g(x,y)} = 1$$

Circle with center $(0,0)$, $r=1$

$f_{max} = \frac{1}{2}$ at

$\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right), \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ with
$\lambda = \frac{1}{2}$



<u>Def</u>: Consider max $f(x,y) = xy$ with
$$x^2+y^2 = a$$
→ Radius of the circle: Parameter

<u>Max. point</u>: $(x^*(a), y^*(a))$

<u>Max. value</u>: $f(x^*(a), y^*(a)) = f^*(a)$

Ex: $\underline{a=1}$: $x^*(1) = \sqrt{\frac{1}{2}}$, $y^*(1) = \sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

OR: $x^*(1) = -\sqrt{\frac{1}{2}}$, $y^*(1) = -\sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

Result:

$$\boxed{\lambda = \frac{d f^*(a)}{d a}}$$

Lagrange multiplier

Change in max value when a changes

small change in a

Interpretation of λ: λ is the marginal change in the max (min) value per unit change in the constant a in the constraint $g(x,y)=a$.

Ex: USE THIS TO APPROXIMATE OPTIMAL VALUES:

$\underline{a=2}$:

$$f^*(2) \approx f^*(1) + \Delta a \underbrace{\frac{d f^*(a)}{d a}}_{=\lambda \text{ from Result}}$$

Example $a=1$ + Result

$$= \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= \underline{\underline{1}}$$

Def. of derivative:

$$f'(a) \approx \frac{\Delta f(a)}{\Delta a}$$

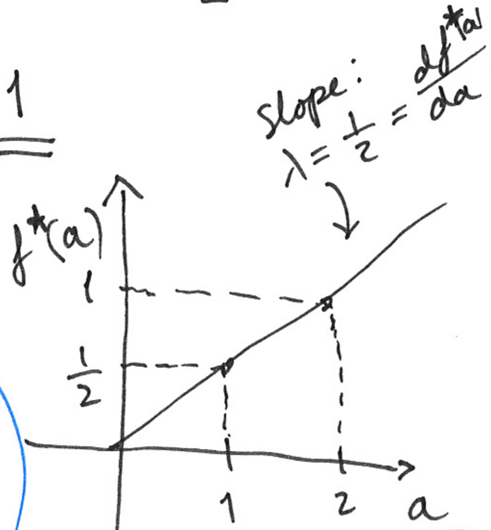
Actually: $f'(a) = \lim_{\Delta a \to 0} \frac{\Delta f(a)}{\Delta a}$

$$\Delta a \, f'(a) \approx \Delta f(a)$$

$$\Delta f^*(a) \approx \Delta a \, (f^*)'(a)$$

$$f^*(2) - f^*(1) \approx (2-1) \underbrace{\frac{d f^*(a)}{d a}}_{=\lambda} $$

Result



Ex ctd: max $f(x,y) = xy$ when $\underbrace{x^2+y^2=a}$

Circle, center $(0,0)$, $r=\sqrt{a}$, $a>0$

EVT: • $f(x,y)$ continuous? Yes.

• Compact constraint set?

$\rightarrow$ Closed? Yes $(=)$

$\rightarrow$ Bounded? Yes (circle)

$\Rightarrow$ EVT holds! The problem has a max (and a min).

• Type ii) points: Admissible points with Degenerate constraint? $g(x,y) = x^2+y^2$

$g'_x = 2x = 0 \Rightarrow x = 0$

$g'_y = 2y = 0 \Rightarrow y = 0$

Then: $x^2+y^2 = 0^2+0^2 = 0 \neq a$, $a>0$

No admissible points with degenerate constraint.

• Type i) points: $\mathcal{L}(x,y) = \underbrace{xy - \lambda(x^2+y^2-a)}_{f-\lambda(g-a)}$

FOC:

(1): $\mathcal{L}'_x = y - \lambda \cdot 2x = 0 \Rightarrow y = 2\lambda x$

(2): $\mathcal{L}'_y = x - \lambda \cdot 2y = 0 \Rightarrow x - \lambda \cdot 2 \cdot 2\lambda x = 0$

C: c: $x^2+y^2=a$

$x - 4\lambda^2 x = 0$

$x(1-4\lambda^2) = 0$

3 cases:

$\underline{x = 0}$:

From (1):
$$y = 2\lambda \cdot 0 = 0$$

From C:
$$0^2 + 0^2 = a$$
$$0 = a$$

For $a=0$: $\to f = 0$

$(0, 0; \lambda)$ → can be anything

If $a \neq 0$: No candidate point

$\underline{\lambda^2 = \frac{1}{4}}$:

$\underline{\lambda = \frac{1}{2}}$:

From (1): $y = 2 \cdot \frac{1}{2} \cdot x = x$

From C: $y^2 + y^2 = a$

$$2y^2 = a$$

$$x = y = \pm\sqrt{\frac{a}{2}}$$

Candidates: $\to f = \frac{a}{2}$

$\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; \frac{1}{2}\right)$

and

$\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; \frac{1}{2}\right) \to f = \frac{a}{2}$

$\underline{\lambda = -\frac{1}{2}}$:

From (1):
$$y = 2\left(-\frac{1}{2}\right)x = -x$$

From C:
$$y^2 + y^2 = a$$
$$2y^2 = a$$
$$y = \pm\sqrt{\frac{a}{2}}$$

Candidates:

$\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; -\frac{1}{2}\right) \to f = -\frac{a}{2}$

and

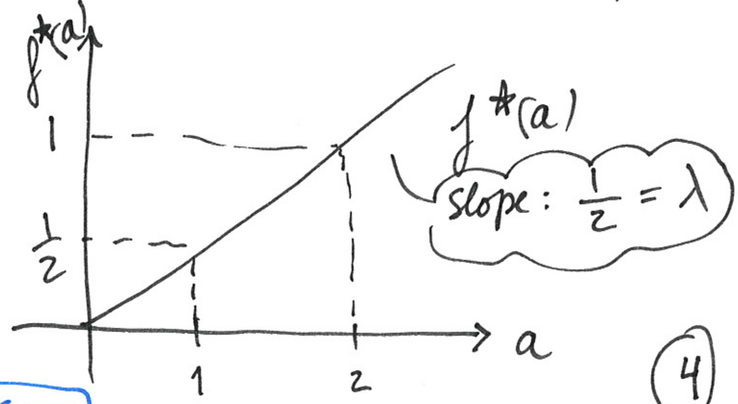
$\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; -\frac{1}{2}\right) \to f = -\frac{a}{2}$

Conclusion: $f_{max} = \frac{a}{2}$ at the max. points $\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$  and $\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$.

$$f^*(a) = \frac{a}{2} = \underbrace{\frac{1}{2}}_{\lambda} a$$

$$\boxed{\frac{d f^*(a)}{d a} = \frac{1}{2} = \lambda}$$

RESULT



Kuhn - Tucker problems

→ Optimization problems with closed inequality constraints ($\leq$, $\geq$).

Ex: $\boxed{\max\ f(x,y) = x^2y^2 \text{ when } x^2+y^2+x^2y^2 \leq 3}$

Kuhn - Tucker problem

Candidate points

(1) Boundary points: Boundary ⇒ = constraint

⇒ $\boxed{\text{Lagrange problem}}$ Solve via standard Lagrange

(2) Interior points: Stationary or other interior critical points of f.

Ex: $f'_x = 2xy^2 = 0 \Rightarrow x=0$ or $y=0$

$f'_y = 2x^2y = 0 \Rightarrow x=0$ or $y=0$

Candidates: $x=0$: INTERIOR POINTS:

- $(0,y)$ when $0^2+y^2+0^2y^2 < 3$

$y^2 < 3$

$-\sqrt{3} < y < \sqrt{3}$

$f(0,y) = 0^2y^2 = 0$

$\underline{y=0}$:

- $(x,0)$ when $x^2+0^2+x^2 0^2 < 3$

$$x^2 < 3$$

$$-\sqrt{3} < x < \sqrt{3}$$

$f(x,0) = x^2 \cdot 0^2$
$= 0$

Lagrange problem / The boundary

max $f(x,y) = x^2y^2$ when $x^2+y^2+x^2y^2=3$

Type i): $\mathcal{L} = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

FOC: $\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0$ (1):

$\mathcal{L}'_y = 2x^2y - \lambda(2y + 2yx^2) = 0$ (2):

C: $x^2+y^2+x^2y^2 = 3$ (3):

From (1): $2x(y^2 - \lambda - \lambda y^2) = 0$ → $x=0$ or $y^2-\lambda-\lambda y^2=0$

From (2): $2y(x^2 - \lambda - \lambda x^2) = 0$ → $y=0$ or $x^2-\lambda-\lambda x^2=0$

Check all combinations:

a) $x=0, y=0$: From (3): $0^2+0^2+0^2 0^2=3$

NOT TRUE $\Rightarrow$ No candidates.

b) $x=0, x^2-\lambda-\lambda x^2=0$: $0^2-\lambda-\lambda\cdot 0^2=0$ $\Rightarrow \lambda=0$

From (3): $0^2+y^2+0^2y^2=3 \Leftrightarrow y^2=3$

$y=\pm\sqrt{3}$

Candidates: $(0, \sqrt{3}; 0)$, $(0, -\sqrt{3}; 0)$

$f=0$ $f=0$

c) $y^2-\lambda-\lambda y^2=0, y=0$: Online. $f=0$

d) $y^2-\lambda-\lambda y^2=0, x^2-\lambda-\lambda x^2=0$: Online.

Candidates: $(1,1;\frac{1}{2})$, $(1,-1;\frac{1}{2})$,

$(-1,1;\frac{1}{2})$, $(-1,-1;\frac{1}{2})$

For all: $\boxed{f=1}$

Type ii: Admissible pts. with degenerate constraint: None. Online.

Conclude: $f_{max}=1$ at $(1,1), (1,-1), (-1,1), (-1,-1)$