

(look at Lagrange problem part of the Kuhn Tucker problem)

Ex: $\max f(x,y) = x^2 y^2$ when $x^2 + y^2 + x^2 y^2 = 3$

$$L = x^2 y^2 - \lambda (x^2 + y^2 + x^2 y^2 - 3)$$

CANDIDATE POINTS:

i) Stationary points of L:

$$L'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0 \quad (1)$$

$$L'_y = x^2 \cdot 2y - \lambda(2y + x^2 \cdot 2y) = 0 \quad (2)$$

$$C: L'_\lambda = x^2 + y^2 + x^2 y^2 = 3 \quad (3)$$

2 kinds
i) Stationary pts. of L
ii) Degenerate constraint
(From Theorem)

From (1):

$$2x(y^2 - \lambda - \lambda y^2) = 0 \quad \begin{cases} x=0 \\ \text{OR} \\ y^2 - \lambda - \lambda y^2 = 0 \end{cases}$$

From (2):

$$2y(x^2 - \lambda - \lambda x^2) = 0 \quad \begin{cases} y=0 \\ \text{OR} \\ x^2 - \lambda - \lambda x^2 = 0 \end{cases}$$

(Check all combinations:)

a) $x=0, y=0$: From (3): $0^2 + 0^2 + 0^2 \cdot 0^2 = 3$; Impossible!
 \Rightarrow No candidates.

b) $x=0, x^2 - \lambda - \lambda x^2 = 0$: $0^2 - \lambda - 0 = 0 \Rightarrow \lambda = 0$

From (3): $0^2 + y^2 + 0^2 y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

So: Candidates: $(0, \sqrt{3}; 0), (0, -\sqrt{3}; 0)$

$$f(0, \sqrt{3}) = 0$$

$$f(0, -\sqrt{3}) = 0$$

$$(c) \underline{y=0, y^2 - \lambda - \lambda y^2 = 0} :$$

$$0^2 - \lambda - \lambda \cdot 0^2 = 0 \Rightarrow \lambda = 0$$

From (3):

$$x^2 + 0^2 + x^2 \cdot 0^2 = 3 \Rightarrow x^2 = 3, \text{ so } x = \pm \sqrt{3}$$

Candidates: $(0, \sqrt{3}; 0), (0, -\sqrt{3}; 0)$

$$d) \underline{y^2 - \lambda - \lambda y^2 = 0, x^2 - \lambda - \lambda x^2 = 0} :$$

$$\rightarrow y^2 = \lambda(1+y^2) \Rightarrow \lambda = \frac{y^2}{1+y^2} \rightarrow \text{never 0}$$

$$x^2 = \lambda(1+x^2) \Rightarrow \lambda = \frac{x^2}{1+x^2} \quad (\star) \rightarrow \text{never 0}$$

$$\Rightarrow \lambda = \lambda \quad \frac{y^2}{1+y^2} = \frac{x^2}{1+x^2} \quad | \cdot (1+y^2)(1+x^2)$$

$$y^2(1+x^2) = x^2(1+y^2)$$

$$y^2 + \cancel{y^2 x^2} = x^2 + \cancel{x^2 y^2}$$

$$y^2 = x^2$$

$$y = \pm x \quad (\star)$$

From (3):

$$x^2 + x^2 + x^2 x^2 = 3$$

$$x^4 + 2x^2 - 3 = 0, \text{ let } u = x^2;$$

$$u^2 + 2u - 3 = 0; \text{ a quadratic eqn:}$$

$$u = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{16}}{2} = \frac{-2 \pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

So $x^2 = 1$ or ~~$x^2 = -3$~~

NOT POSSIBLE!

$$x = \pm 1$$

Hence, $y = \pm x = \pm 1 \rightarrow$ OBS! 4 combinations:

From (*)

$x = 1:$
 $y = 1$ $y = -1$

$x = -1:$
 $y = 1$ $y = -1$

Also, $\lambda = \frac{x^2}{1+x^2} = \frac{1}{2} \Rightarrow$

From (*)

Candidates: $(1, 1; \frac{1}{2}), (1, -1; \frac{1}{2}),$
 $(-1, 1; \frac{1}{2}), (-1, -1; \frac{1}{2})$

} (*)

For all of these: $f = 1$

$x^2 + y^2$
 (sign irrelevant due to square)

ii) Admissible points with degenerate constraints:

$$g(x, y) = x^2 + y^2 + x^2 y^2 = 3$$

$$g'_x = 2x + 2xy^2 = 0 \Rightarrow 2x(1 + y^2) = 0 \Rightarrow x = 0$$

Never 0

$$g'_y = 2y + x^2 2y = 0 \Rightarrow 2y(1 + x^2) = 0 \Rightarrow y = 0$$

Never 0

$\Rightarrow (x, y) = (0, 0)$; But this is not admissible since then

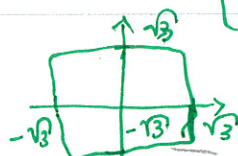
$g(0, 0) = 0^2 + 0^2 + 0^2 0^2 = 0 \neq 3$, so the constraint doesn't hold.

Hence, no candidates of type ii)

Conclusion: $f_{\max} = 1$ at $(1, 1), (1, -1), (-1, 1)$
 and $(-1, -1)$ with $\lambda = \frac{1}{2}$

Is there a max?
D: $x^2 + y^2 + x^2 y^2 = 3$
 • Closed? ✓
 • Bounded? ✓
 Note that x^2, y^2 and $x^2 y^2 > 0 \forall x, y$. Hence, both x^2 and y^2 must be ≤ 3 .
 But then:
 $x^2 \leq 3$ and $y^2 \leq 3 \Leftrightarrow$
 $-\sqrt{3} \leq x \leq \sqrt{3}$
 $-\sqrt{3} \leq y \leq \sqrt{3}$

the admissible x 's and y 's fit into this square



So D is bounded $\Rightarrow D$ is compact & f cont. \Rightarrow EVT holds $\Rightarrow \exists$ max and min. (13)