EBA 1180
Sect. 46
Spring 24

Warm-up:

Q: • What does the Lagrange thm. say?

• What kind of points are candidates for max/min in a Lagrange problem?

Lagrangian:
$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda(g(x,y) - a)$

Thm: If (x^*, y^*) is max/min in a Lagrange problem:

$$\boxed{\max/\min\ f(x,y) \text{ with } g(x,y) = a}$$

Then either

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfies the Lagrange constraints FOC + C:

FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

OR:

ii) The constraint is degenerate at (x^*, y^*), i.e.:

$g'_x = 0$ and $g'_y = 0$ and $g(x,y) = a$

More Lagrange problems

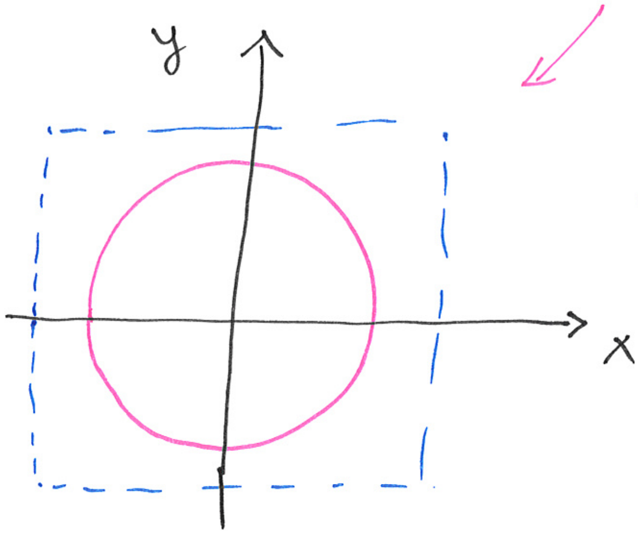
Ex: max/min $f(x,y) = xy$ when $\underbrace{x^2+y^2}_{g(x,y)}=1$

NOTE:

$D: x^2+y^2=1$ is compact:

closed ✓ (=)

bounded ✓



Circle, center $(0,0)$, $r=\sqrt{1}=1$

$\Downarrow$

EVT: f has a max and min over D since it is continuous.

→ Degenerate constraint? $g'_x = 2x = 0 \Rightarrow x=0$

$g'_y = 2y = 0 \Rightarrow y=0$, but

then: $x^2+y^2 = 0^2+0^2 = 0 \neq 1$, so the constraint doesn't hold (point not admissible).

Hence, there are no admissible points with degenerate constraint $\Rightarrow$ No type ii) candidates (ref. Thm).

NB: Holds in general for circles.

Lagrangian: $\mathcal{L}(x,y) = xy - \lambda\,(x^2+y^2-1)$

FOC:

$$\mathcal{L}'_x = y - \lambda \cdot 2x = 0$$

$$\mathcal{L}'_y = x - \lambda \cdot 2y = 0$$

C: $x^2+y^2=1$

3 equations with 3 unknowns: x, y, λ

$$y = 2\lambda x$$

$$x - \lambda \cdot 2(2\lambda x) = 0$$

$$x(1-4\lambda^2) = 0$$

$x = 0$:

$$y = 2\lambda x = 2\lambda \cdot 0 = 0$$

$$x^2+y^2 = 0^2+0^2 = 0 \neq 1$$

so C: doesn't hold

$\Rightarrow$ Not a candidate point.

$$1-4\lambda^2=0$$

$$\lambda^2 = \frac{1}{4}$$

$\lambda = \frac{1}{2}$:

$$y = 2\cdot\frac{1}{2}x = x$$

C: $x^2+y^2=1$

$$2x^2=1$$

$$x^2=\frac{1}{2}$$

$$x = \pm\sqrt{\frac{1}{2}} = y$$

$\lambda = -\frac{1}{2}$:

$$y = 2\left(-\frac{1}{2}\right)x = -x$$

C: $x^2+y^2=1$

$$2x^2=1$$

$$x^2=\frac{1}{2}$$

$$x = \pm\sqrt{\frac{1}{2}}$$

$$y = -x$$

Candidate points:

$\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$ → $f(x,y) = xy = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}$

$\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ → $\boxed{\text{MAX}}$

$f(x,y) = xy = \left(-\sqrt{\tfrac{1}{2}}\right)\left(-\sqrt{\tfrac{1}{2}}\right) = \tfrac{1}{2}$

Candidate points:

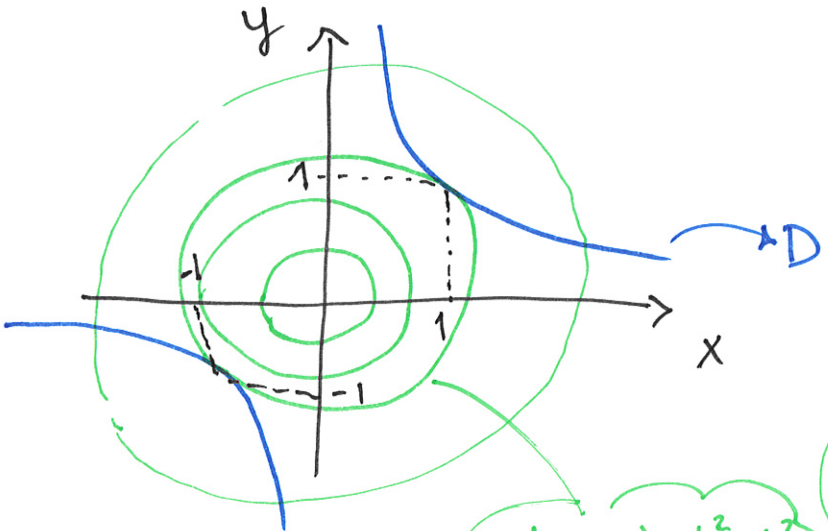
$\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ → $f(x,y) = xy = \sqrt{\tfrac{1}{2}}\left(-\sqrt{\tfrac{1}{2}}\right) = -\tfrac{1}{2}$

$\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$ → $\boxed{\text{MIN}}$

$f(x,y) = xy = \left(-\sqrt{\tfrac{1}{2}}\right)\sqrt{\tfrac{1}{2}} = -\tfrac{1}{2}$

Ex: max/min $f(x,y) = x^2 + y^2$ when $\overbrace{xy = 1}^{D}$

$y = \frac{1}{x}$



Level curves for f:

$f(x,y) = x^2 + y^2 = c$

$f(x,y) = 1^2 + 1^2 = 2$

EVT? Closed (=) ✓

Bounded? NO!

Hence, D is not compact ⇒ Can't use EVT.

$$\mathcal{L}(x,y) = x^2 + y^2 - \lambda\,(xy - 1)$$

<u>FOC:</u>

$$\mathcal{L}'_x = 2x - \lambda y = 0$$

$$\mathcal{L}'_y = 2y - \lambda x = 0$$

<u>C:</u>

$$xy = 1$$

3 eqns, 3 <u>unknowns:</u>

x, y, λ

$$2x = \lambda y$$

$$x = \frac{\lambda y}{2}$$

$$2y - \lambda\left(\frac{\lambda y}{2}\right) = 0 \quad \Big|\cdot 2$$

$$4y - \lambda^2 y = 0$$

$$y\,(4 - \lambda^2) = 0$$

<u>3 cases:</u>

<u>$y = 0$:</u>

$$x = \frac{\lambda \cdot 0}{2} = 0$$

<u>C:</u> $xy = 0 \cdot 0 = 0 \neq 1$

Not a candidate pt. because the constraint doesn't hold.

<u>$\lambda = 2$</u>

$$x = \frac{2y}{2} = y$$

<u>C:</u> $xy = 1$

$$x^2 = 1$$

$$x = \pm 1$$

<u>Candidate pts:</u>

$(1, 1) \rightarrow f = 1^2 + 1^2 = 2$

and

$(-1, -1) \rightarrow f = 2$

<u>$\lambda = -2$:</u>

$$x = -y$$

<u>C:</u> $xy = 1$

$$-y^2 = 1$$

$$\underbrace{y^2 = -1}$$

NOT POSSIBLE!

$\Downarrow$

No candidate points!

Admissible points with degenerate constraint?

$\boxed{g(x,y) = xy}$

$$g'_x = y = 0$$
$$g'_y = x = 0 \quad \Rightarrow \quad C: \; xy = 0 \cdot 0 = 0 \neq 1$$

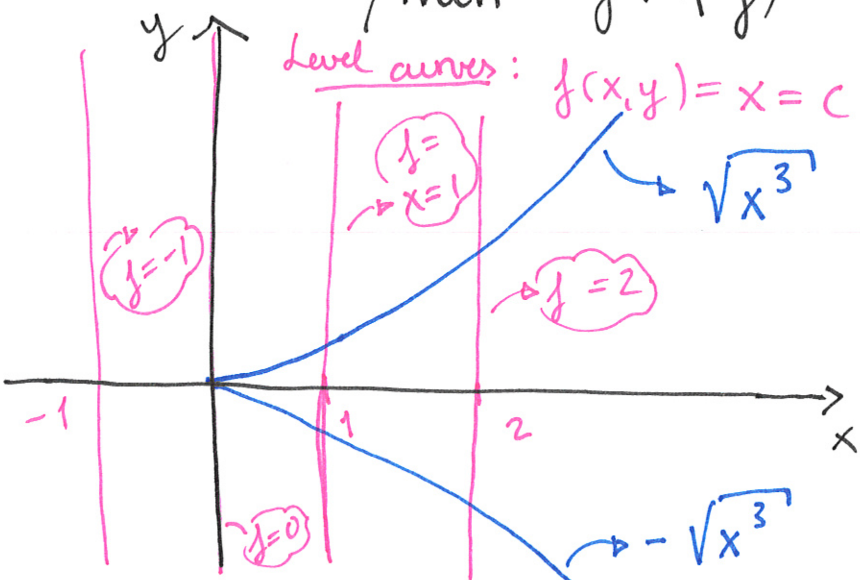
$\Rightarrow$ There are no admissible points with a degenerate constraint.

CONCLUSION: $f_{min} = 2$ at $(1,1)$ and $(-1,-1)$ with $\lambda = 2$.

No maximum since $y = \frac{1}{x}$ will satisfy the constraint. Can let $x \to \infty$. Then, $y \to 0$, but is admissible. Then

$$f(x,y) = x^2 + y^2 \to \text{"} \infty^2 + 0^2 \text{"} = \infty .$$

Ex: max/min $f(x,y) = x$ when $\underbrace{y^2 - x^3 = 0}$



D: $y^2 - x^3 = 0$

$y = \pm\sqrt{x^3}$

NB: Only def. for $x^3 \geq 0$

$\Rightarrow x \geq 0$

Can we have admissible point with a degenerate constraint?

$$g'_x = -3x^2 = 0 \implies x = 0$$

$$g'_y = 2y = 0 \implies y = 0 \text{, but then}$$

$$g(x,y) = g(0,0) = 0^2 - 0^3 = 0 \text{, so}$$

$(0,0)$ is on D. Hence, $(0,0)$ is an admissible point with a degenerate constraint.

Do we have any ordinary (type i) candidate points?

$$\mathcal{L}(x,y) = x - \lambda(y^2 - x^3)$$

FOC:

$$\mathcal{L}'_x = 1 + \lambda \cdot 3x^2 = 0 \quad : (1)$$

$$\mathcal{L}'_y = -\lambda \cdot 2y = 0 \quad : (2)$$

C:

$$y^2 - x^3 = 0 \quad : (3)$$

(2): $-\lambda \cdot 2y = 0$

$\lambda = 0$:

(1): $1 + 0 \cdot 3x^2 = 0$

$1 = 0$

NOT TRUE $\implies$ No candidate point!

$y = 0$:

(3): $0^2 - x^3 = 0 \implies x = 0$

(1): $1 + \lambda \cdot 3 \cdot 0 = 0$

$1 = 0$

NOT TRUE! $\implies$ No candidate.

Hence, we have no ordinary candidate points.

All we have: Admissible point with degenerate constraint; $(0,0)$.

From the Figure: We see that this is the minimum point. No maximum (from figure OR no candidates).

D intersects level curves with larger and larger value

OR: $y = \pm\sqrt{x^3}$ makes constraint hold. Can choose arbitrarily large $x \Rightarrow f(x,y) = x \longrightarrow \infty \Rightarrow$ NO MAX