Lagrange problems

EBA 1180
Spring 24
Lect. 45

- Lagrange problems = optimization problems (max/min) with equality constraints.

$$(\star) \quad \boxed{\max/\min\ f(x,y) \quad \text{when} \quad \underbrace{g(x,y)}_{\text{function}} = \underbrace{a}_{\text{constant}}}$$

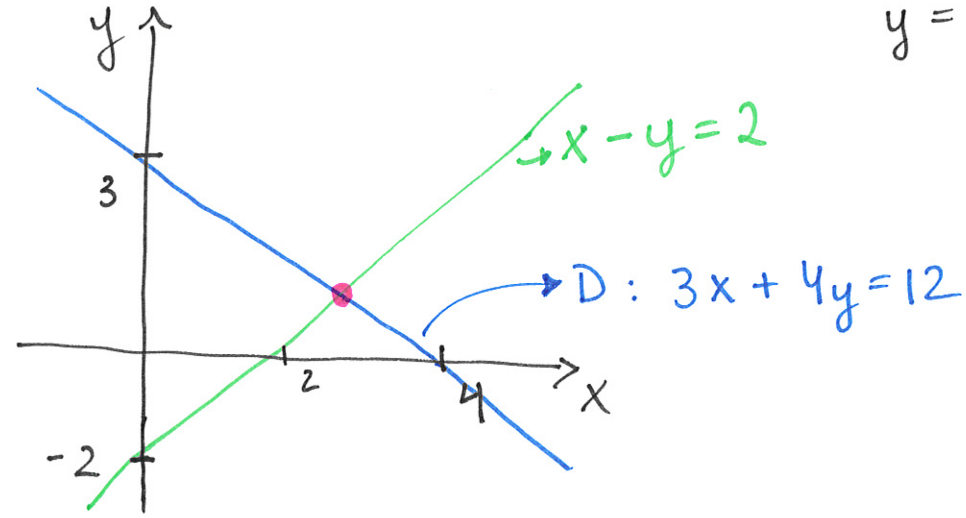
Ex: $\min f(x,y) = x^2 + y^2$ when $\underbrace{3x+4y}_{g(x,y)} = \underbrace{12}_{a}$

Draw: $4y = 12 - 3x \quad | :4$

$y = 3 - \frac{3}{4}x$;

$x = 0: \quad y = 3$

$y = 0: \quad x = 4$



Ex: $\min f(x,y) = x^2 + y^2$ when $3x + 4y = 12$

and $x - y = 2$ → $y = x - 2$

} A point: Intersection of the two str. lines

Recall: General method:

1) Find candidate points

2) Determine wheter any of these are max/min.

For Lagrange problems:

i) Interior stationary pts: NONE

ii) Other interior critical pts: NONE

iii) Boundary points

Extreme value theorem:
If D is compact (closed and bounded) and f is continuous, then f has a max/min on D.

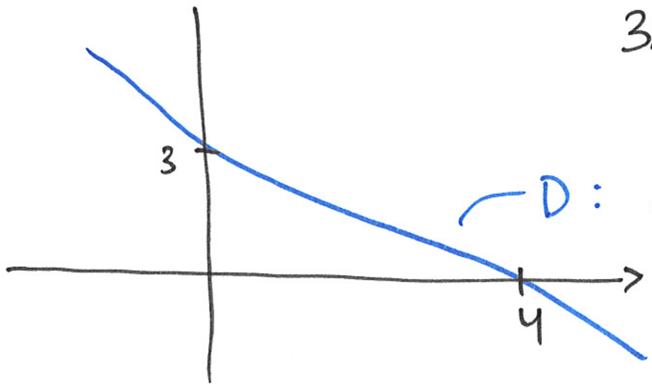
and

Always true for Lagrange problems: = constraint

Not necessarily true for Lagrange problems

Ex: $\min f(x,y) = x^2 + y^2$ when

$3x + 4y = 12$



D: $3x + 4y = 12$; D is closed, but not bounded $\Rightarrow$

D is not compact $\Rightarrow$ EVT can't be used.

Method of Lagrange multipliers

NB: $= 0$ if constraint in (*) holds

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\,(g(x,y) - a)$$

Lagrangian (Lagrange function)

Lagrange multiplier $\rightarrow$ variable

$$= x^2 + y^2 - \lambda(3x + 4y - 12)$$

Candidates for max/min: The stationary points of $\mathcal{L}$:

EXAMPLE

First order conditions

$$\text{FOC: } \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 = 2x - 3\lambda \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 = 2y - 4\lambda \end{cases}$$

$$\text{C: } \{ \ \mathcal{L}'_\lambda = -(g(x,y) - a) = 0 = -(3x + 4y - 12) = 0$$

Constraint

$$g(x,y) - a = 0 \qquad \Updownarrow$$

$$g(x,y) = a \qquad 3x + 4y = 12$$

The constraint

Lagrange conditions: FOC + C

FOC:

$$\mathcal{L}'_x = 2x - 3\lambda = 0 \quad :(1)$$

$$\mathcal{L}'_y = 2y - 4\lambda = 0 \quad :(2)$$

C:

$$3x + 4y = 12 \quad :(3)$$

System of 3 eqns and 3 unknowns: x, y, λ

From (1): $2x = 3\lambda \Rightarrow x = \frac{3}{2}\lambda$ (*)

From (2): $2y = 4\lambda \Rightarrow y = 2\lambda$ (**)

From (3): $3 \cdot \frac{3}{2}\lambda + 4 \cdot 2\lambda = 12 \quad | \cdot 2$

$$9\lambda + 16\lambda = 24$$

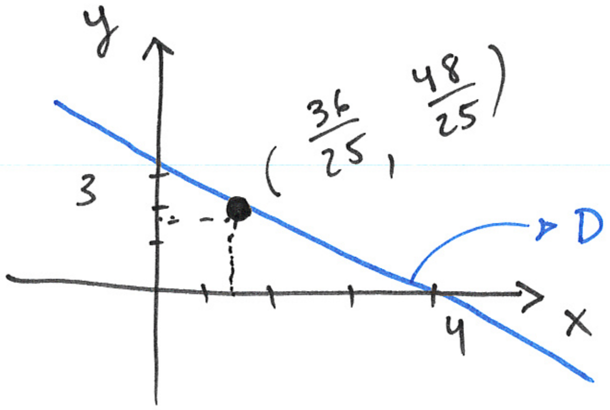
$$25\lambda = 24$$

$$\lambda = \frac{24}{25}$$

$\Rightarrow (\star)$: $$x = \frac{3}{2} \cdot \frac{24}{25} = \frac{3 \cdot 12}{25} = \underline{\frac{36}{25}}$$

$(\star\star)$: $$y = 2 \cdot \frac{24}{25} = \underline{\frac{48}{25}}$$

Only one candidate point : $\left(\underbrace{\frac{36}{25}}_{x}, \underbrace{\frac{48}{25}}_{y}; \underbrace{\frac{24}{25}}_{\lambda}\right)$



Alternative method (substitution)

$$\boxed{\min f(x,y) = x^2 + y^2 \text{ when } 3x + 4y = 12}$$

$4y = 12 - 3x$

$$y = 3 - \frac{3}{4}x$$

$$x^2 + y^2 = x^2 + \left(3 - \frac{3}{4}x\right)^2$$

$$= x^2 + 9 - \frac{9}{2}x + \frac{9}{16}x^2$$

Define a one-var. function $g(x)$

$$= \frac{25}{16}x^2 - \frac{9}{2}x + 9 =: g(x)$$

$$\boxed{\text{Alt: } \min g(x) = \frac{25}{16}x^2 - \frac{9}{2}x + 9}$$

A one variable optimization problem

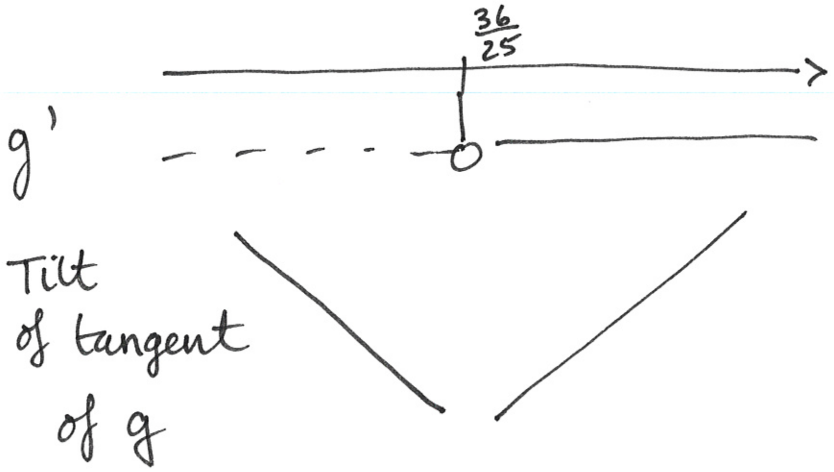
$$g'(x) = \frac{25}{16}\cdot 2x - \frac{9}{2} = 0$$

$$\frac{25}{8}x - \frac{9}{2} = 0 \qquad \Big| \cdot 8$$

$$25x - 36 = 0$$

$$x = \frac{36}{25}$$

$$y = 3 - \frac{3}{4}\cdot\frac{36}{25} = \dots = \frac{48}{25}$$



Hence, $x = \frac{36}{25}$ is a minimum for $\underline{\underline{g.}}$

Intuition: Lagrange multiplier method

$\underline{\text{Ex}}$: max/min $f(x,y) = x^2 + y^2$ when $3x + 4y = 12$

Level curves of f: $f(x,y) = c$

$x^2 + y^2 = c$ $\rightarrow$ Circle, center $(0,0)$, $r = \sqrt{c}$, $c > 0$.

If $c = 0$: A point, $(0,0)$

No points if $c < 0$

$\underline{c=1}$: $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$

$\underline{c=2}$: $x^2 + y^2 = 2$, $r = \sqrt{2}$

$\underline{c=4}$: $x^2+y^2=4$, $r=\sqrt{4}=2$

$\underline{c=9}$: $x^2+y^2=9$, $r=\sqrt{9}=3$

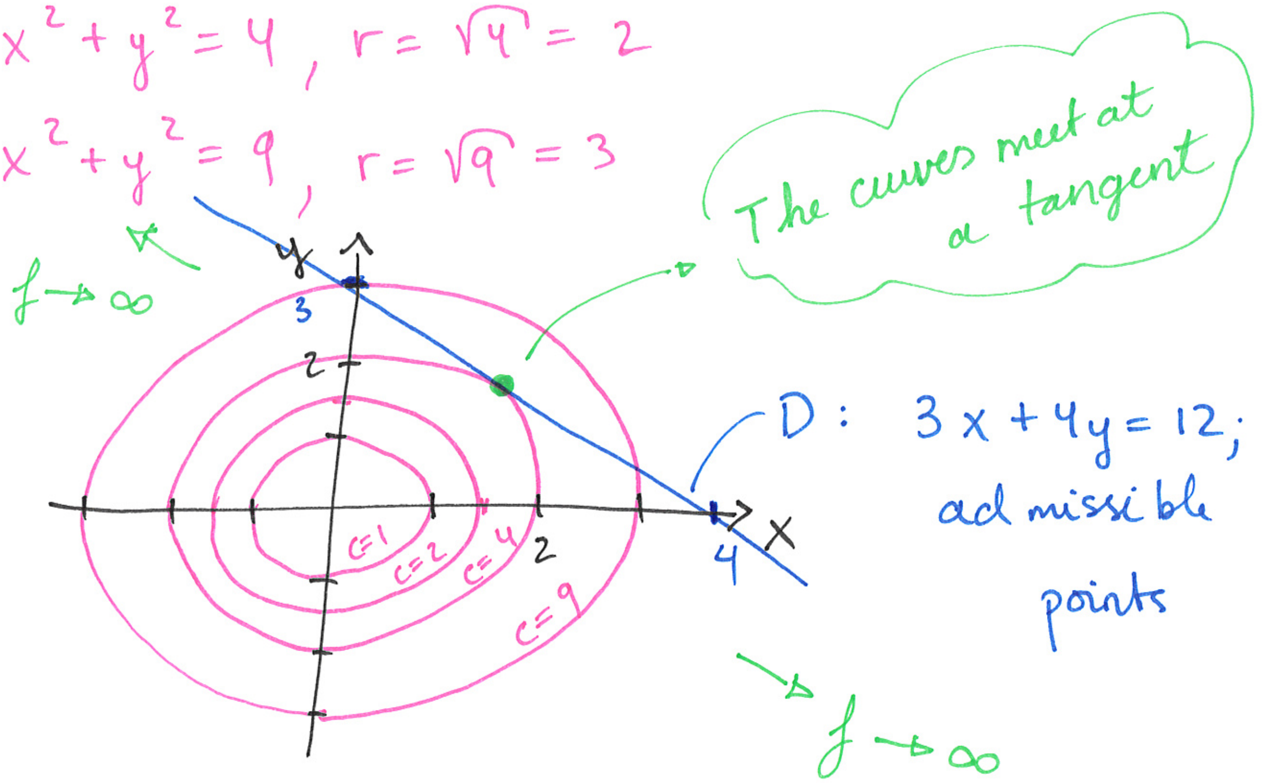
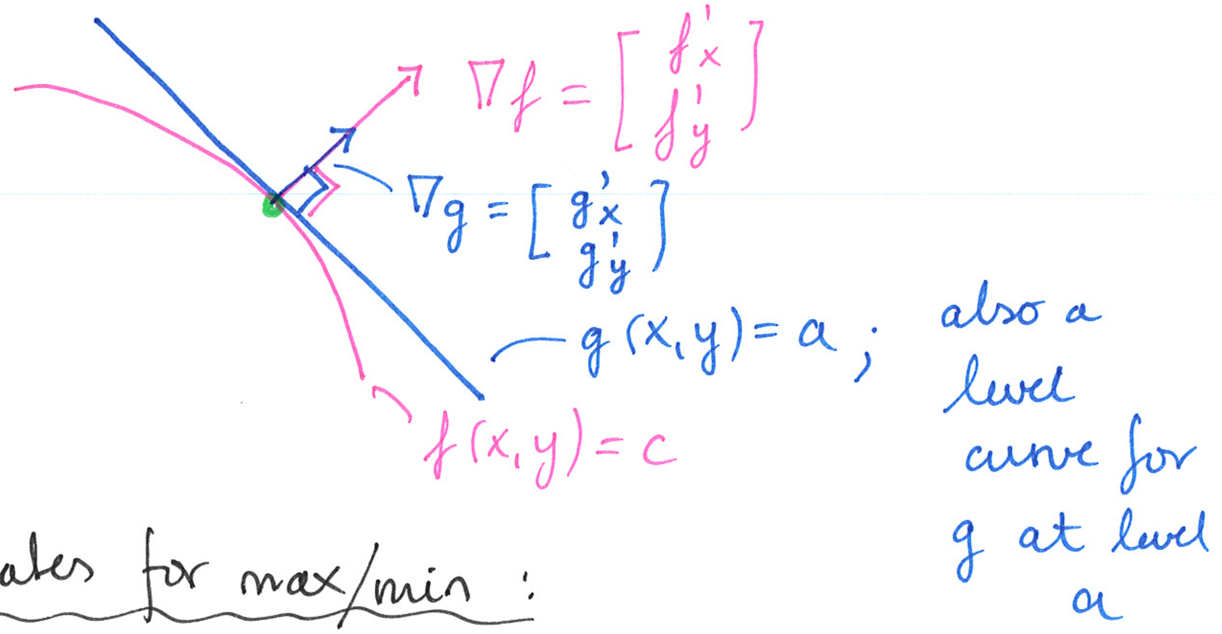


FIG. 1 :



Candidates for max/min :

Points where the two curves meet at a tangent

$$\begin{cases} 3x+4y=12 & : D \\ x^2+y^2=c & : \text{Level curve of } f \end{cases}$$

Slopes of the tangents of level curves should be equal :

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

→ See notes on tangents of level curves: Implicit differentiation

$$\cancel{-}\frac{2x}{2y} = \cancel{-}\frac{3}{4}$$

$$4x = 3y$$

$$y = \frac{4}{3}x$$

Constraint: $$3x + 4\left(\frac{4}{3}x\right) = 12 \quad \Big| \cdot 3$$

$$9x + 16x = 36$$

$$25x = 36$$

$$\underline{x = \frac{36}{25}}$$

NOTE:

From FIG. 1

$$\nabla f = \lambda \nabla g$$

Gradient of f is a scalar multiple of the gradient of g

$$\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \lambda \begin{bmatrix} g'_x \\ g'_y \end{bmatrix}$$

$$\begin{cases} f'_x = \lambda g'_x \\ f'_y = \lambda g'_y \end{cases} \Rightarrow \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 \end{cases}$$

Theorem: If (x^*, y^*) is max/min in a Lagrange problem:

$$\boxed{\max/\min f(x,y) \text{ with } g(x,y) = a}$$

Then, either:

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfies the Lagrange constraints FOC + C:

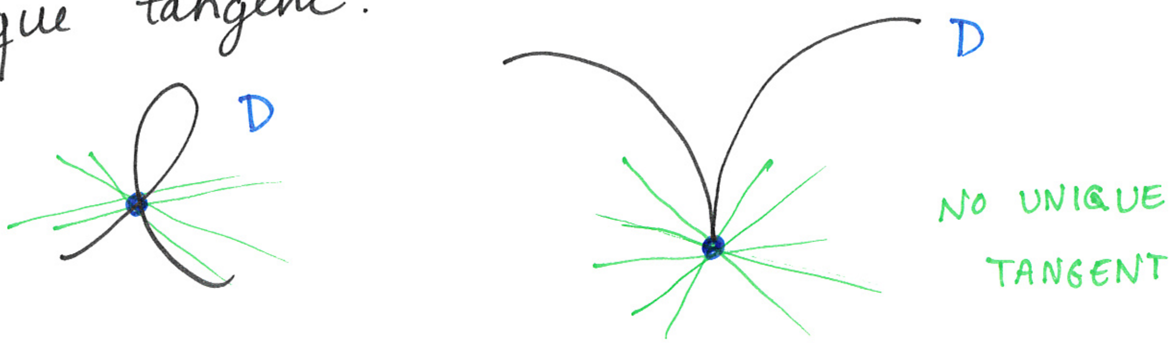
FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

OR

ii) The constraint is degenerate at (x^*, y^*), i.e.;

$g'_x = 0$ and $g'_y = 0$ and $g(x,y) = a$

Ex: In general, an extreme pt. with a degenerate constraint is a point where D does not have a unique tangent.



Ex: min x^2+y^2 where $\underbrace{3x+4y}_{g(x,y)}=12$

$g'_x = 3 \neq 0$

$g'_y = 4 \neq 0$, so case ii) of Theorem is not possible.

$\boxed{B2-070}$