Examples: Optimization

EBA1180
Sect. 44
Spring 24

Warm-up: $f(x,y) = x^2y^3 + y^2 - 2y$,

$D_f = \mathbb{R}^2$

No boundary

An unconstrained optimization problem

Q: max/min $f(x,y)$?

In general:

Candidate pts:

i) Stationary pts.

ii) Other Critical pts.

iii) Boundary

CLASSIFY CANDIDATES:

2nd derivative test

i) Stationary points:

$$\begin{cases} f'_x = 2xy^3 = 0 \Rightarrow x=0 \text{ and/or } y=0 \\ f'_y = 3x^2y^2 + 2y - 2 = 0 \end{cases}$$

$3\cdot 0^2\cdot 0^2 + 2\cdot 0 - 2 = 0$

$-2 = 0$

NOT TRUE

$\Downarrow$

$(0,0)$ not possible

2 cases

$\underline{x=0}$:

$3\cdot 0^2\cdot y^2 + 2y - 2 = 0$

$2y = 2$

$y = 1$

$\underline{y=0}$:

$3x^2\cdot 0^2 + 2\cdot 0 - 2 = 0$

$-2 = 0$

NOT TRUE;

$(x,0)$ not possible

Hence, only one stationary point $(x^*, y^*) = (0,1)$.

Note: $f(0,1) = 0^2\cdot 1^3 + 1^2 - 2\cdot 1 = \underline{\underline{-1}}$

ii) Other critical pts: Partial derivatives are defined everywhere $\Rightarrow$ No.

iii) Boundary? No.

$\Rightarrow$ The stationary point is the only candidate point.

Classification of stationary point

f''_{xx} $\quad$ f''_{xy}

$$H(f)(x,y) = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{bmatrix}$$

Hessian matrix $\quad$ f''_{yx} $\quad$ f''_{yy}

Insert $(0,1)$:

$$H(f)(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ so}$$

$$\det H(f)(0,1) = 2 \cdot 2 - 0 \cdot 0 = 4 > 0,$$

$$\operatorname{tr} H(f)(0,1) = 2 + 2 = 4 > 0$$

$\Downarrow$ 2nd derivative test

$f(0,1) = -1$ is a local minimum.

NOTE: If $x = 1$, then:

$$f(1,y) = \underbrace{y^3 + y^2 - 2y}$$

(the y^3 will overpower $y^2 - 2y$)

Can be made arbitrarily small (test on calculator if unsure: $y = -100, -1000, -10\,000$)

OR be made arbitrarily large (try: $y = 100, 1000, 10\,000$)

(The $y^3 + y^2$ will overpower $-2y$)

$\Downarrow$

f has no (global) max or (global) min.

NB:
Example of a function with only one stationary pt, local min, but still no global minimum.

Exercise sheet 43

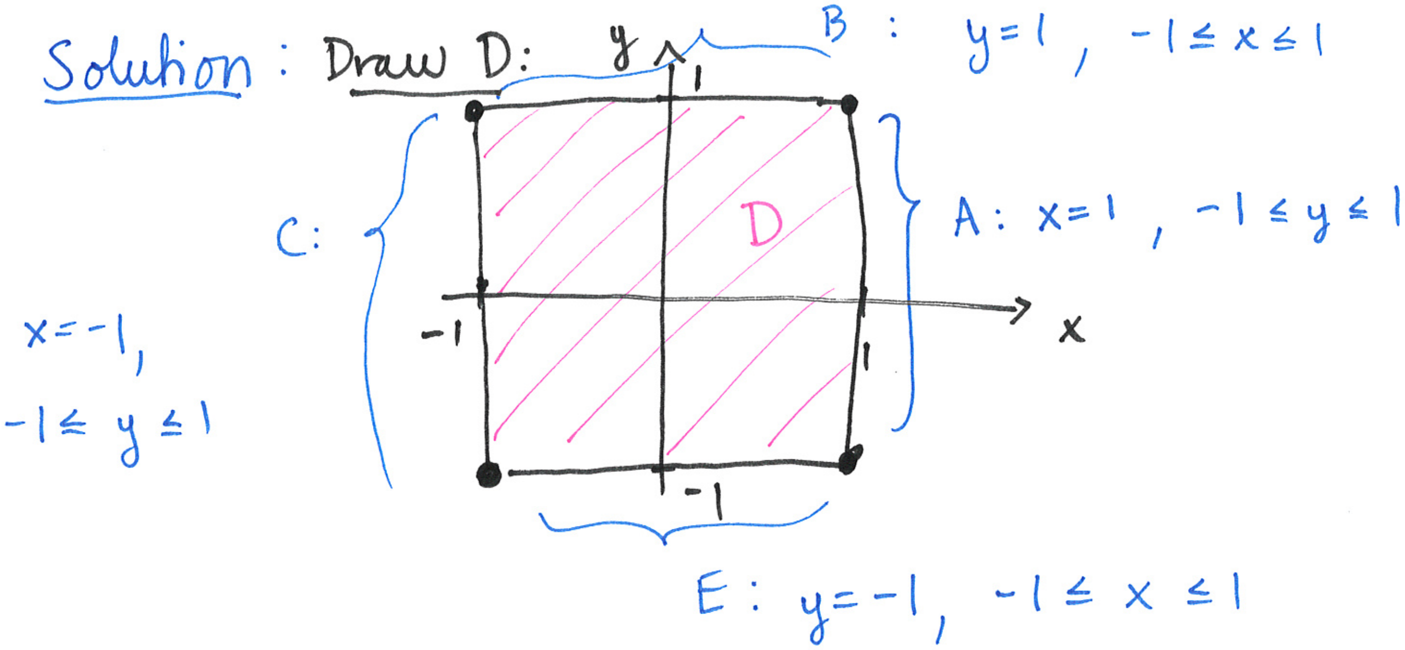
5) d) max/min $f(x,y) = xy(x^2 - y^2)$

$$= x^3y - xy^3, \text{ when}$$

$$\underbrace{-1 \leq x, y \leq 1}_{D}$$

($-1 \leq x \leq 1$ and $-1 \leq y \leq 1$)

Solution: Draw D:



NB: f is continuous, D is closed ($\leq$) and bounded (can be boxed in) $\Rightarrow$ D is compact

$\Rightarrow$ The extreme value theorem gives f has a max. and min. (over D).

Candidate points

i) Interior stationary points:

$$f'_x = 0 \Rightarrow f'_x = 3x^2y - y^3 = 0$$
$$f'_y = 0 \Rightarrow f'_y = x^3 - 3xy^2 = 0$$

(1) $y\cdot(3x^2 - y^2) = 0$

(2) $x(x^2 - 3y^2) = 0$

From (1):

$\underline{y=0}$: or $\underline{3x^2 - y^2 = 0}$: $3x^2 = y^2$

(2): $x(x^2 - 3\cdot 0^2) = 0$

$x^3 = 0$

$\underline{x = 0}$

(2):

$x(x^2 - 3\cdot 3x^2) = 0$

$x(x^2 - 9x^2) = 0$

$-8x^3 = 0$

$x = 0$

$\Downarrow$

$y^2 = 3\cdot 0^2 = 0$

$y = 0$

Only one interior stationary point: $(x,y) = (0,0)$

$\Rightarrow \underline{f(0,0) = 0}$

ii) Other interior critical points: No other such points since f'_x and f'_y are defined everywhere.

iii) Boundary of D :

$\underline{A}$: $\underline{x=1, \ -1 \le y \le 1}$:

$:=$; define

$f(x,y) = f(1,y) = y - y^3 =: h(y)$

$h'(y) = 1 - 3y^2 = 0$

$$3y^2 = 1$$

$$y^2 = \frac{1}{3}$$

$$y = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}} \quad \left(\text{Between } -1 \text{ and } 1\right)$$

Candidates from A: $\left(1, -\frac{1}{\sqrt{3}}\right)$, $\left(1, \frac{1}{\sqrt{3}}\right)$, $(1,1)$, $(1,-1)$

$f\left(1, -\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = -\frac{2}{3\sqrt{3}}$

$f(1,-1) = 0$

$f(1,1) = 1 - 1^3 = 0$

Boundary pts. of A viewed as a one variable function: y-axis

$f\left(1, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}}$

B: $y = 1$, $-1 \le x \le 1$:

$$f(x,1) = x(x^2-1) =: h(x)$$
$$= x^3 - x$$

$$h'(x) = 3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

Candidates from B: $\left(-\frac{1}{\sqrt{3}}, 1\right), \left(\frac{1}{\sqrt{3}}, 1\right),$

$(1,1), (-1,1)$

$f(-1,1) = \underline{0}$

$f(1,1) = \underline{0}$

Boundary pts. of B

$$f\left(-\frac{1}{\sqrt{3}}, 1\right) = -\frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \underline{\frac{2}{3\sqrt{3}}}$$

$$f\left(\frac{1}{\sqrt{3}}, 1\right) = \underline{-\frac{2}{3\sqrt{3}}}$$

C: $x = -1, \ -1 \le y \le 1$:

$$f(-1, y) = y^3 - y =: h(y)$$

$$h'(y) = 3y^2 - 1 = 0$$

$$y = \pm \frac{1}{\sqrt{3}}$$

$$f\left(-1, -\frac{1}{\sqrt{3}}\right) = \underline{\frac{2}{3\sqrt{3}}}$$

Candidates from C: $\left(-1, -\frac{1}{\sqrt{3}}\right), \left(-1, \frac{1}{\sqrt{3}}\right),$

$(-1, 1), (-1,-1)$

$f(-1,-1) = \underline{0}$

Boundary pts of C

$f(-1,1) = \underline{0}$

$$f\left(-1, \frac{1}{\sqrt{3}}\right) = \underline{-\frac{2}{3\sqrt{3}}}$$

$\underline{E:}$ $\underbrace{y=-1,\ -1 \le x \le 1}$:

$$f(x,-1) = x - x^3 =: h(x)$$

$$h'(x) = 1 - 3x^2 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$f(-\frac{1}{\sqrt{3}}, -1) = \underline{\frac{-2}{3\sqrt{3}}}$

Candidates from E: $(-\frac{1}{\sqrt{3}}, -1)$, $(\frac{1}{\sqrt{3}}, -1)$, $(-1,-1)$, $(1,-1)$

$f(-1,-1) = \underline{0}$ $\quad f(1,-1) = \underline{0}$ $\quad f(\frac{1}{\sqrt{3}}, -1) = \underline{\frac{2}{3\sqrt{3}}}$

To conclude: Compare function values of all candidates $(4 \cdot 4 = 16)$. Know min/max (over D) from EVT. Hence,

Maximum: $$f_{max} = \frac{2}{3\sqrt{3}}$$

at max points: $(1, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, 1)$, $(-1, -\frac{1}{\sqrt{3}})$, $(\frac{1}{\sqrt{3}}, 1)$.

Minimum: $$f_{min} = -\frac{2}{3\sqrt{3}}$$

at min points: $\left(1, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -1\right)$.