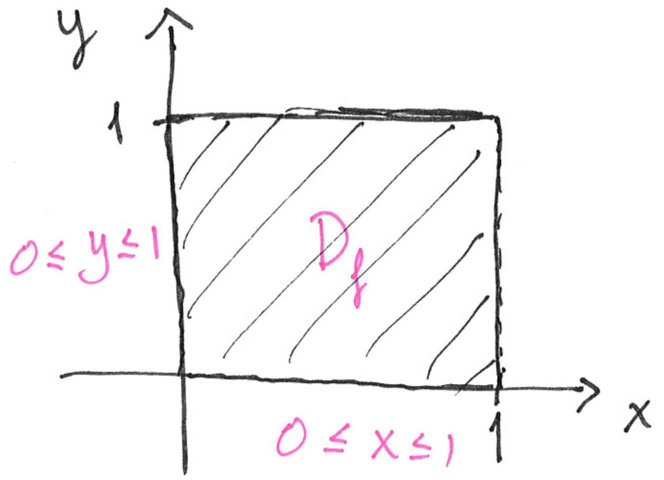
EBA 1180
Lect. 43
Spring 24

Warm up: $f(x,y) = x + y$, $\underbrace{0 \le x, y \le 1}_{D_f}$

What are the (global) max/min points?



See directly:

Maximum: $(1,1) \Rightarrow$

$f(1,1) = 2$

Minimum: $(0,0) \Rightarrow$

$f(0,0) = 0$

NOTE: f has both (global) min and max.

Q1: What if D_f : $0 < x, y < 1$

Q2: What if min/max $x+y$ over all of $\mathbb{R}^2$?

Constrained optimization and the extreme value theorem

- $f(x,y)$ is a continuous function on a set D in $\mathbb{R}^2$.

Extreme value theorem (EVT): If f is a continuous function on a compact set D in $\mathbb{R}^2$, then f has a maximum and a minimum on D.

Compact sets

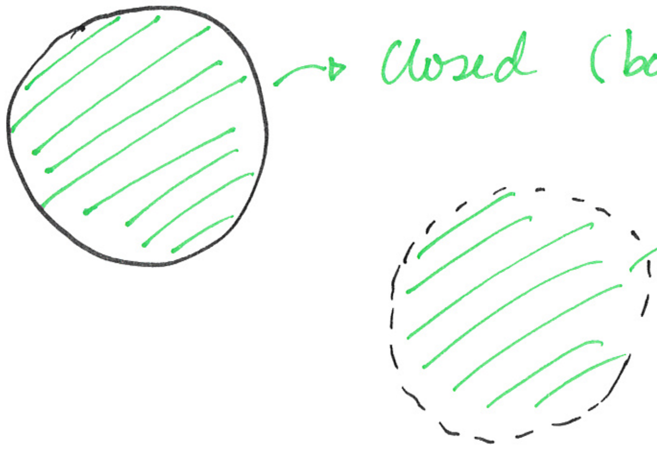
Def (Compact set): A subset D of $\mathbb{R}^2$ is compact if it is closed and bounded.

Def (Closed set): A subset D of $\mathbb{R}^2$ is closed if all boundary points of D are included in D.

NOTE: $=$, $\leq$, $\geq$; Closed

$<$, $>$; Not closed

Ex:

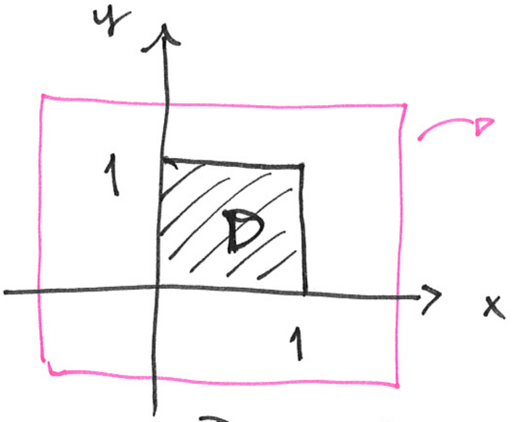


Closed (boundary included)

Not closed (boundary not included)

Def (Bounded set): A subset D of $\mathbb{R}^2$ is bounded if there exists a rectangle in $\mathbb{R}^2$ (with finite side lengths) that includes all of D.

Ex:



Rectangle containing D

$D: \ 0 \le x, y \le 1$

$\Rightarrow$ Bounded ✓

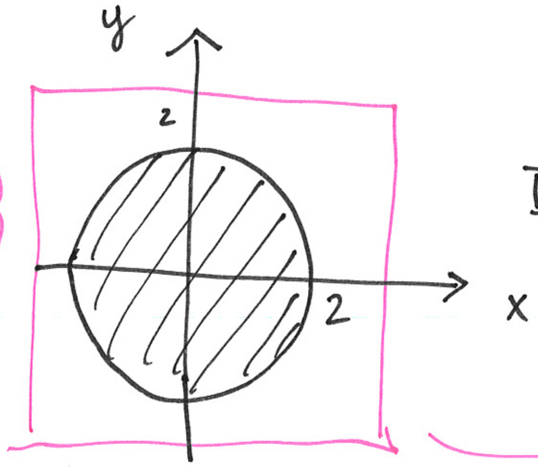
Closed ✓

$\Downarrow$

Compact ✓

Ex:

Closed? Bounded? Compact?



Q:

$D: \ x^2 + y^2 \le 4$

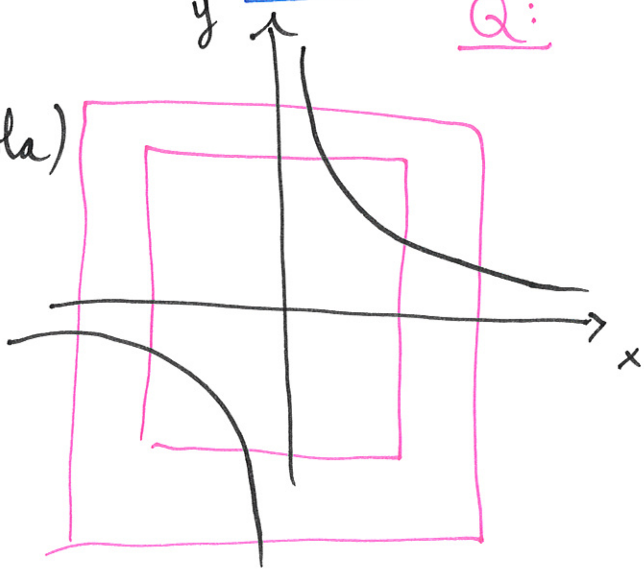
Closed? ✓ (includes boundary)

Bounded? ✓

$\Downarrow$

$\boxed{\text{Compact}}$

Q:



Ex: $D: \ xy = 1$ (hyperbola)

$y = \frac{1}{x}$

Closed? ✓

Bounded? NO!

Constrained optimization

$$\text{max/min } \underbrace{f(x,y) = x^2+y^2}_{\text{Objective function}} \text{ when } \underbrace{0 \le x, y \le 1}_{\text{Constraints}}$$

Constrained optimization

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}$, subset of $\mathbb{R}^2$.

Set of admissible points

∞

max/min $f(x,y) = x^2 + y^2$

Unconstrained optimization

max/min $f(x,y)$ when (x,y) in D

Unconstrained

max/min $f(x,y)$

Candidate points

i) Stationary pt: $f'_x = 0$, $f'_y = 0$

ii) Other critical pts: f'_x or f'_y are not defined.

iii) Boundary points of D_f

- (Local) classification:

Second derivative test (local max, local min, saddle pt.)

- Must check: Are any of these global max/min?

Constrained

Candidate pts:

i) Interior stationary pt: $f'_x = 0$, $f'_y = 0$

ii) Other interior critical points: f'_x ~~or~~ f'_y not defined.

iii) Boundary points of D; ∂D The boundary of D

$D = \{(x,y) : \text{all constraints satisfied}\}$

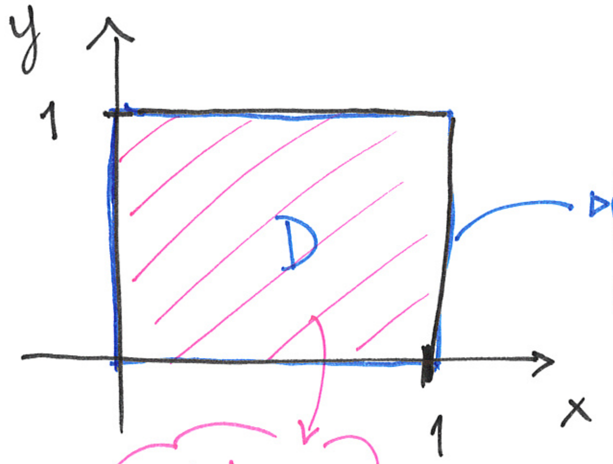
EVT: If D is compact (closed and bounded), there is global max/min.

- Determine if candidate pts. are global max/min: Use EVT if D is compact.

The boundary: ∂D

Ex: max/min $f(x,y) = x^2 + y^2$ when

$0 \le x, y \le 1$



∂D = the boundary of D (the four sides of the square)

Interior of D

Max: Make x and y as large as possible $\Rightarrow$

$x = y = 1 \Rightarrow f(1,1) = 2$

Q:

Min: Make x and y as small as possible $\Rightarrow$

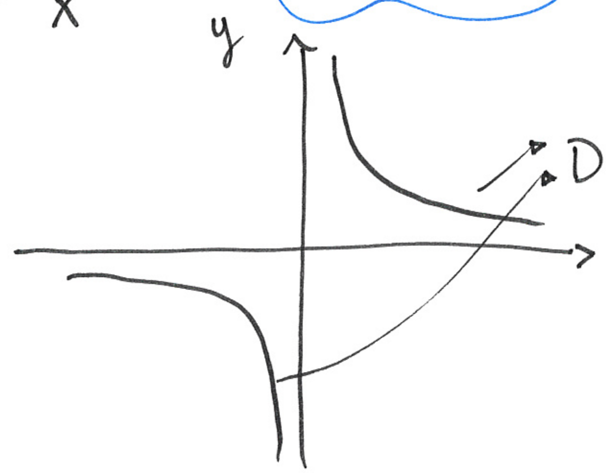
$x = y = 0 \Rightarrow f(0,0) = 0$

Ex: D:

$xy = 1$

$y = \frac{1}{x}$

$x=0$ impossible since $xy=1$



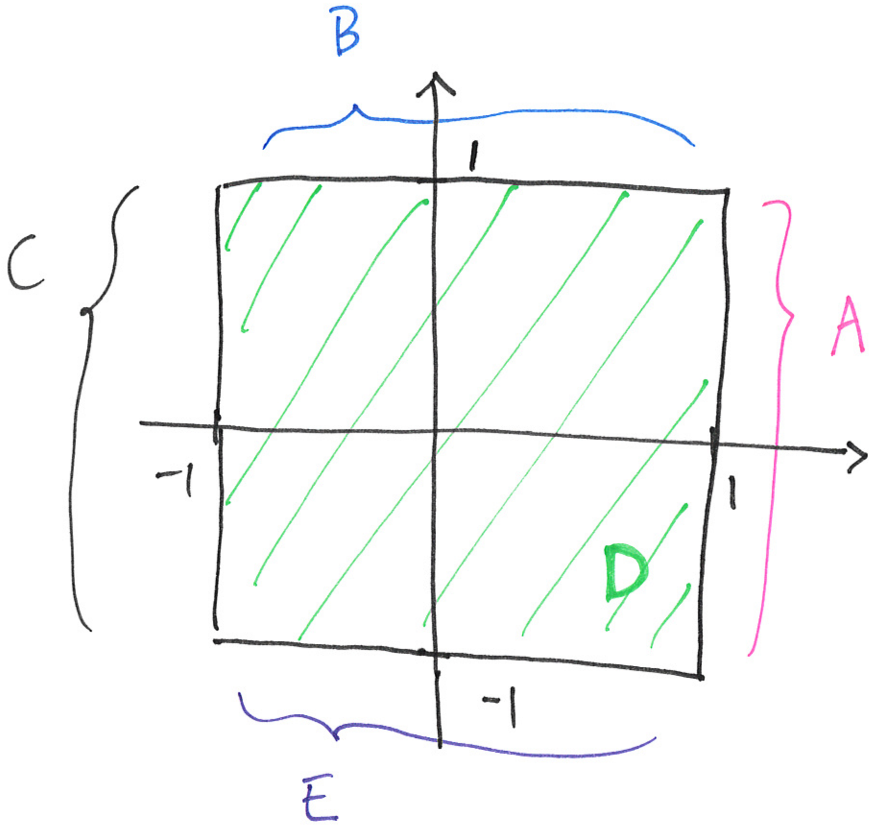
$D = \{ (x,y) \in \mathbb{R}^2 : xy = 1 \}$

∂D = boundary of D

= all pts. on D.

An example: Constrained optimization

<u>Ex</u>: max/min $f(x,y) = x^2 + y^2$ when $-1 \leq x, y \leq 1$



<u>Candidate pts</u>:

i) Interior stationary point:

$f'_x = 2x = 0 \Rightarrow x = 0$

$f'_y = 2y = 0 \Rightarrow y = 0$

$(0,0)$ is an interior pt. of D.

Candidate: $(x,y) = (0,0) \Rightarrow f(0,0) = 0$

ii) <u>Other interior critical points</u>: None.

iii) <u>Boundary points</u>: ∂D = four sides of square

$$\begin{cases} A: x = 1, \; -1 \leq y \leq 1 \\ B: y = 1, \; -1 \leq x \leq 1 \\ C: x = -1, \; -1 \leq y \leq 1 \\ E: y = -1, \; -1 \leq x \leq 1 \end{cases}$$

EVT:
- f continuous? OK!
- D compact? Closed and bounded?
  Yes! ✓ ✓

$\Rightarrow$ $\boxed{f(x,y) \text{ has (global) max and min}}$

EVT holds

$\Downarrow$

There is a max and min among the candidate points.

What is the max/min? Compare values of candidate points.

i) Stationary: $(0,0) \Rightarrow f(0,0) = 0$

ii) Critical: None.

$f(x,y) = x^2 + y^2$

iii) Boundary:

A: $f(1,y) = 1 + y^2$, $-1 \le y \le 1$, see dir.:

max: $f(1,1) = f(1,-1) = 2$,

min: $f(1,0) = 1$

B: $f(x,1) = x^2 + 1$, $-1 \le x \le 1$. See dir:

max: $f(1,1) = f(-1,1) = 2$, min: $f(0,1) = 1$

C: $f(-1,y) = 1 + y^2$, max: $f(-1,1) = f(-1,1) = 2$

min: $f(-1,0) = 1$

E: $f(x,-1) = x^2 + 1$, $-1 \le x \le 1$

max: $f(1,-1) = f(-1,-1) = 2$

min: $f(0,-1) = 1$

Conclusion: D is compact, so there is a max/min from EVT. The ~~largest~~ highest function value among the candidate points is:

$f_{max} = 2$ → global max value

at the max. pts: $(1,1), (1,-1), (-1,1), (-1,-1)$

The lowest value among the candidate pts. is:

$$f_{min} = 0 \to \text{global min. value}$$

at the min. pt. $(0,0)$

Alt. method side A: $f(1,y) = 1+y^2, \ -1 \leq y \leq 1$

$x=1 \Rightarrow$

$(1+y^2)' = 2y = 0 \Rightarrow y=0$

Candidates: $y=0, \ y=-1, \ y=1$