Course paper 2023

EBA 1180
Spring 24
Lect. 42

1) g) $\int_1^{e^2} \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{\sqrt{u}}{\cancel{x}} \cancel{x}\,du$

SUBSTITUTION:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$x = e^2 \Rightarrow u = \ln x = \ln e^2 = 2$

$x = 1 \Rightarrow u = \ln x = \ln 1 = 0$

$= \int_0^2 \sqrt{u}\,du = \int_0^2 u^{\frac{1}{2}}\,du$

$= \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{u=0}^{2} = \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$= \frac{2}{3}\left(2\sqrt{2} - 0\right) = \frac{4}{3}\sqrt{2}$

$2^{\frac{3}{2}} = 2^{1+\frac{1}{2}} = 2^1 \cdot 2^{\frac{1}{2}} = 2\sqrt{2}$

General formula for a parabola

2.) P: $f(x) = a(x-2)^2 + 5$

since $x = 2$ is the axis of symmetry and $y = 5$ is the vertex.

Parabola intersects the x-axis in $x = 2 \pm \sqrt{5}$:

From P:

$$f(2 \pm \sqrt{5}) = 0$$

$$a(\pm\sqrt{5})^2 + 5 = 0$$

$$5a = -5$$

$$\underline{a = -1}$$

P: $f(x) = 5 - (x-2)^2 = \underline{1 + 4x - x^2}$

H: $(x-0)(y-0) = c$ General formula for a hyperbola

since $x = 0$ and $y = 0$ are asymptotes.

$$xy = c$$

$$y = \frac{c}{x}$$

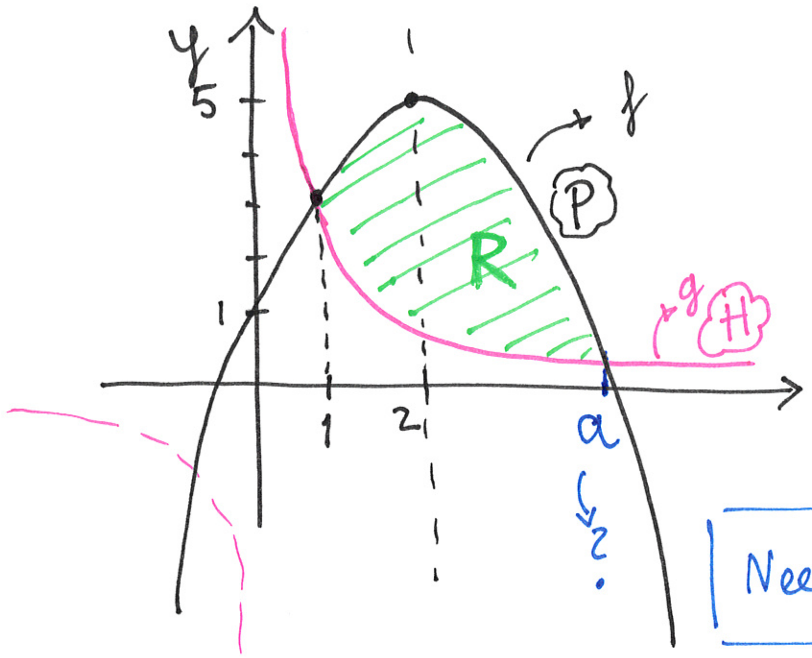
H: $g(x) = \frac{c}{x}$; What should c be?

parabola = hyperbola in $x=1$

Intersection in $x = 1$: $f(1) = g(1)$

$$1 + 4 \cdot 1 - 1^2 = \frac{c}{1} \Rightarrow \underline{c = 4}$$

$$\underline{g(x) = \frac{4}{x}}$$



Need to find a!

b) Area of R $= \int_1^a f(x) - g(x)\, dx$

Find a : Intersection: $f(x) = g(x)$

$$1 + 4x - x^2 = \frac{4}{x} \quad \Big| \cdot x$$

$$x + 4x^2 - x^3 = 4$$

$$x^3 \cancel{+} 4x^2 \cancel{+} x + 4 = 0$$

Minus!

Know intersect at $\boxed{x=1}$, so $f(1) = g(1)$, i.e. eq. holds at $x = 1$

$$(x-1)(x^2 - 3x - 4) = 0$$

$\underline{x = 1}$ or $x^2 - 3x - 4 = 0$

$$(x-4)(x+1) = 0$$

$\underline{x = 4}$, $\underline{x = -1}$

Polynomial ~~equation~~ division

$(x^3 - 4x^2 - x + 4) : (x-1) = x^2 \ldots$

$-(x^3 - x^2)$

...

$\Rightarrow \quad \underline{a = 4}$

FROM FIGURE

$$\text{Area of } R = \int_1^4 \underbrace{1 + 4x - x^2}_{f(x)} - \underbrace{\frac{4}{x}}_{g(x)} \, dx$$

$$= \left[x + 2x^2 - \frac{1}{3}x^3 - 4\ln|x| \right]_{x=1}^{4}$$

$$= \dots \text{ insert numbers} \dots = \underline{\underline{12 - 8\ln 2}}$$

3.) Total cash flow:

$$\int_0^{25} f(t)\,dt = \int_0^{25} 100\, e^{\sqrt{t}}\, dt$$

$$= \int_0^5 100\, e^u\, 2\underbrace{u}_{\sqrt{t}}\, du$$

Substitution:

$u = \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}\, dt$

$$= \int_0^5 200\, e^u\, u\, du$$

$$= 200 \int_0^5 e^u\, u\, du$$

BOUNDS:

$t = 0 \Rightarrow u = \sqrt{t} = \sqrt{0} = 0$

$t = 25 \Rightarrow u = \sqrt{t} = \sqrt{25} = 5$

$$= 200\left[ue^u - e^u\right]_{u=0}^{5}$$

Int. by parts:

$$\int e^u u\,du = e^u u - \int e^u \cdot 1\,du$$

$$\int w' v = wv - \int w v'$$

$w' = e^u \Rightarrow w = e^u$

$v = u \Rightarrow v' = 1$

$$= 200\left(5e^5 - e^5\right) - 200\left(\underbrace{0e^0}_{0} - \underbrace{e^0}_{1}\right)$$

$$= \ldots = \underline{\underline{800e^5 + 200}}$$

Expression for net present value:

$$\int_0^{25} f(t)e^{-rt}\,dt = \int_0^{25} 100e^{\sqrt{t}}e^{-rt}\,dt$$

6) b) $x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 + c\vec{v}_4 = \vec{w}$

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 5 & 3 & 1 & 3 & a \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \end{array}\right] \quad -1$$

TRICK

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \quad \begin{array}{l} -4 \\ -7 \end{array}$$

$-4 *$ row 1, add to row 2

$-7 *$ row 1, add to row 3

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & -12 & 36 & 48 & c-7(a-b) \end{array}\right] \; \cdot\left(-\frac{12}{7}\right) \rightarrow$$

$-\frac{12}{7}$ * row 2, add to row 3

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & 0 & 0 & 0 & \underbrace{c-7(a-b)-\frac{12}{7}(b-4(a-b))}_{(*)} \end{array}\right]$$

$\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \iff$

the lin. system is consistent $\iff (*) = 0$

$$\overbrace{c-7(a-b)-\frac{12}{7}\big(b-4(a-b)\big)}^{(*)} = 0 \quad \Big| \cdot 7$$

$$7c-49(a-b)-12b+48(a-b) = 0$$

$$7c-12b-(a-b) = 0$$

$$-a-11b+7c = 0 \quad \Big| \cdot (-1)$$

$$a+11b-7c = 0$$

Conclusion: $\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

$\iff \quad a+11b-7c = \underline{\underline{0}}.$

7) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

X must be 2×2 for AX and XA to be def.

$$\boxed{AX = XA}$$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

For $AX = XA$:

$$\left.\begin{aligned} c &= b \\ d &= a \\ a &= d \\ b &= c \end{aligned}\right\}$$

c, d free, $a = d$ and $b = c$

$$(a, b, c, d) = (d, c, c, d) = c\,(0, 1, 1, 0) + d\,(1, 0, 0, 1),$$

c and d are free.

Conclusion:

$$X = c\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} + d\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

where c, d are free.

$$= \begin{bmatrix} d & c \\ c & d \end{bmatrix}$$