Graphs and level curves

EBA 1180
Spring 24
Lect. 39

Ex: $f(x,y) = x^2 + y^2$

Level curves? $\to$ $f(x,y) = c$

$c = 2$: $f(x,y) = 2$

$x^2 + y^2 = 2$ $\to$ circle, center $(0,0)$, $r = \sqrt{2}$

$c = 1$: $f(x,y) = 1$

$x^2 + y^2 = 1$ $\to$ circle, center $(0,0)$, $r = 1$

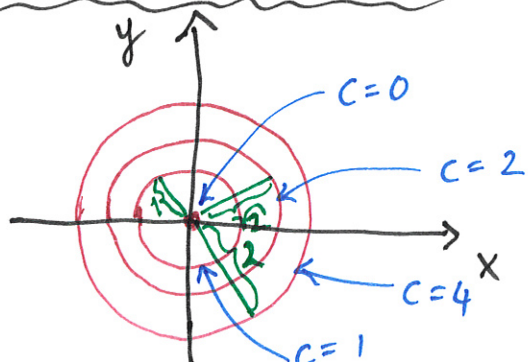
$c = 0$: $f(x,y) = 0$

$x^2 + y^2 = 0 \iff x = y = 0$ level "curve" is just a point

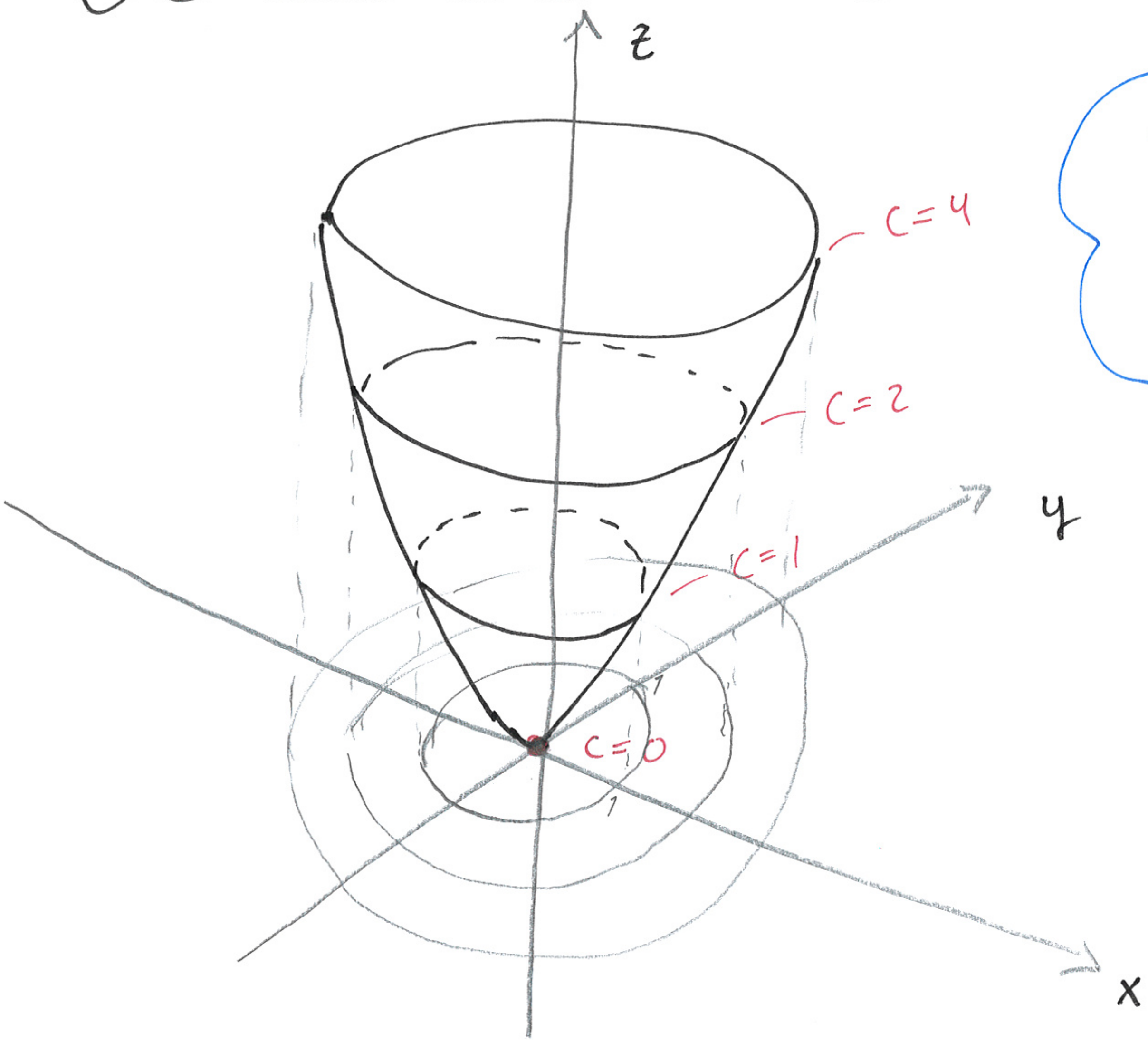
$c = 4$: $x^2 + y^2 = 4$ $\to$ circle, center $(0,0)$, $r = \sqrt{4} = 2$

$c = -1$: $x^2 + y^2 = -1$ $\to$ no such points

Illustration of level curves from above: In xy-plane



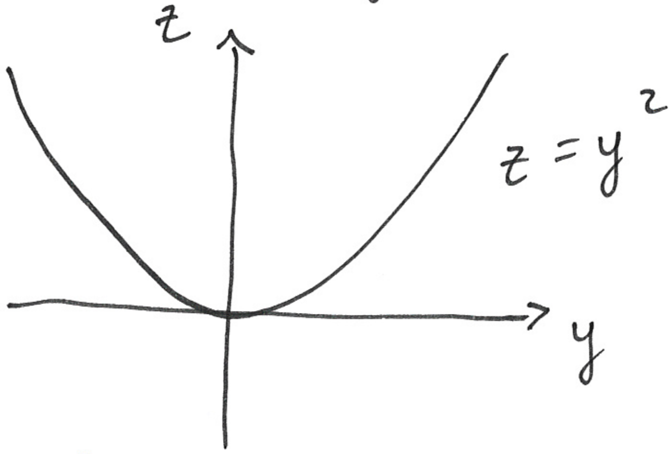
→ Use level curves to draw the graph of f:



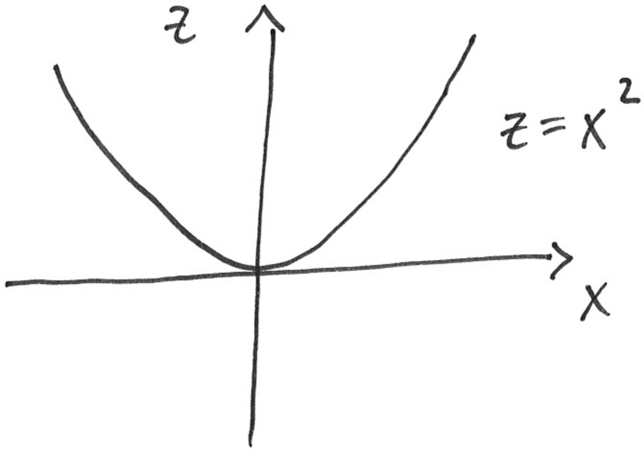
TRICK: Draw surface first, then axes.

Q: If $x=0$, what does $z = f(x,y) = f(0,y)$ look like?

Cut $x=0$: $z = f(0,y) = 0^2 + y^2 = y^2$



Cut $y=0$: $z = f(x,0) = x^2 + 0^2 = x^2$



Linear functions

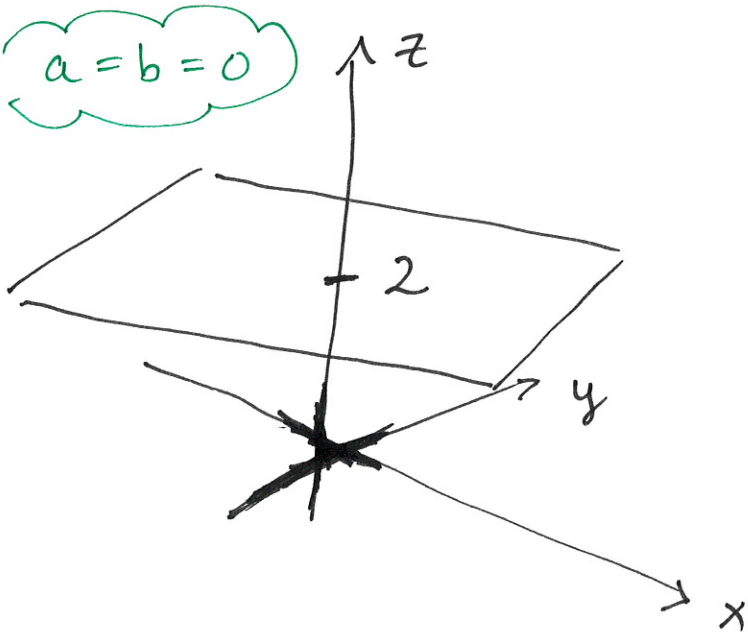
Def (linear function): A function in two variables is linear if it can be written:

$$f(x,y) = ax + by + c$$

FACT: The graph of f is a plane $\Leftrightarrow$ f is linear.

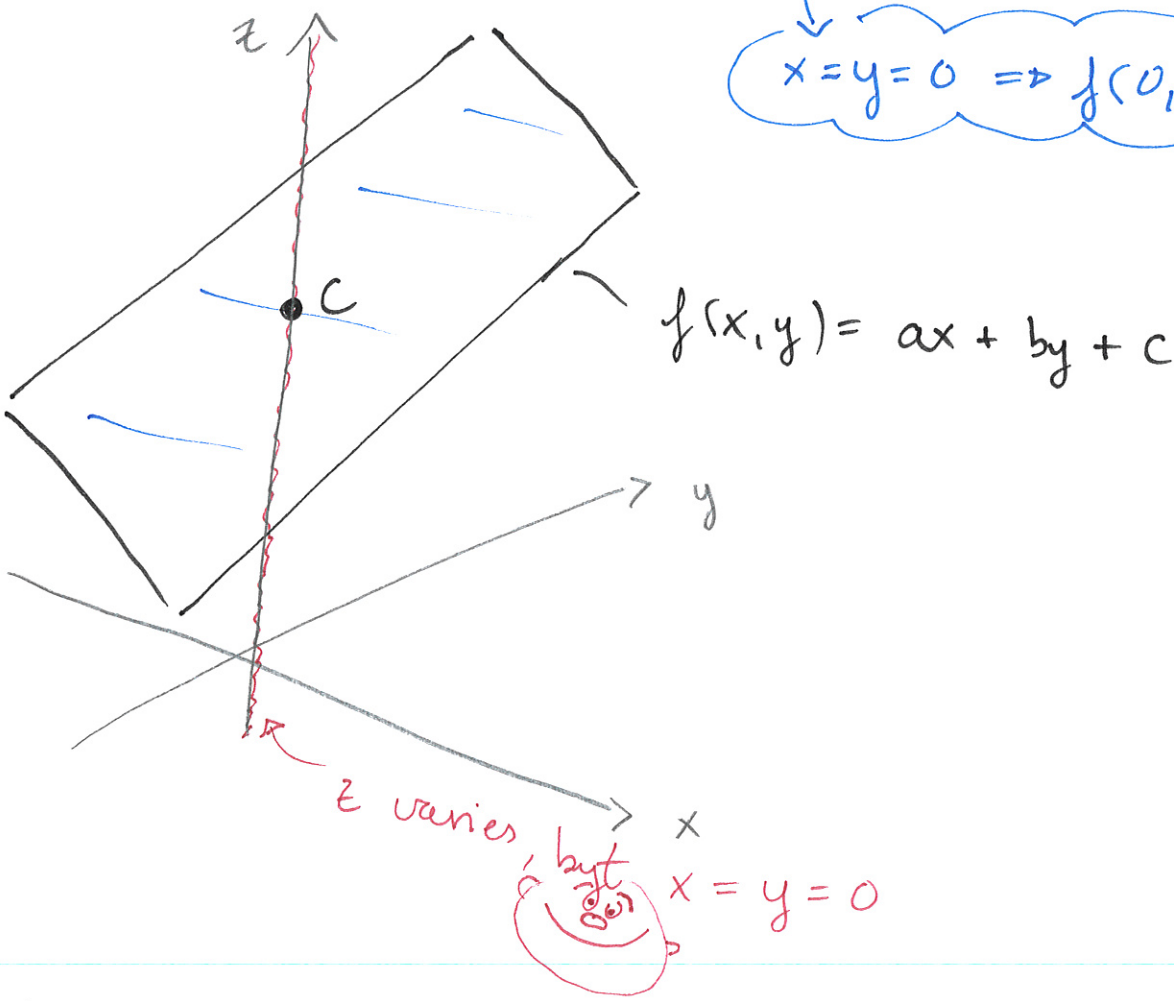
Ex: $f(x,y) = 2$ $\rightarrow$ $a=b=0$

$z = f(x,y) = 2$



NB: The intersection of the graph of $f(x,y) = ax + by + c$ and the z-axis is $z = c$.

$x = y = 0 \Rightarrow f(0,0) = c$



Linear functions with $c = 0$

$$f(x,y) = ax + by$$

$$z = ax + by$$

$$0 = ax + by - z \iff \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \perp \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence, the graph of $f(x,y) = ax + by$: All vectors

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that are normal to $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$

$\perp$, $90°$

This is a plane and $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is its normal vector.

Ex: $f(x,y) = x - 2y$

$z = x - 2y$ $\Rightarrow$ The plane that is the graph of $f(x,y)$ has normal vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

$0 = x - 2y - z$

Conclusion: The graph of a linear function in two variables $f(x,y) = ax + by + c$ is a plane with normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ and intersection with the z-axis $z = c$.

Partial derivatives of functions in two variables

Ex: $f(x,y) = 3x + 4y - 5$

$f(x,y) = x^2 + y^2$

Partial derivatives:

"is defined"

$$f'_x(x,y) := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

"partial derivative of f wrt. x"

TO COMPUTE: Think of y as a constant. Use normal rules of differentiation to f'_x.

$$f'_y(x,y) := \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

COMPUTE: Think of x as a constant.

Ex: i) $f(x,y) = 3x + 4y - 5$

$f'_x(x,y) = 3 + 0 - 0 = \underline{\underline{3}}$

$f'_y(x,y) = 0 + 4 - 0 = \underline{\underline{4}}$

ii) $f(x,y) = x^2 + y^2$

$$f'_x(x,y) = 2x + 0 = \underline{\underline{2x}}$$

$$f'_y(x,y) = 0 + 2y = \underline{\underline{2y}}$$

Double derivatives:

$$f''_{xx}(x,y) = 2 \quad , \quad f''_{yy}(x,y) = 2$$

NOTE: Cross derivatives!

$$f''_{xy}(x,y) = 0 \quad , \quad f''_{yx}(x,y) = 0$$

SAME!

Def (Stationary point): Let $f(x,y)$ be a function. A point $(x,y) = (a,b)$ is a stationary point for f if

$$f'_x(a,b) = 0 = f'_y(a,b)$$

- To find stationary points: Solve the system of eqns

Solve for (x,y)

$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases}$$

The Hessian of $f(x,y)$:

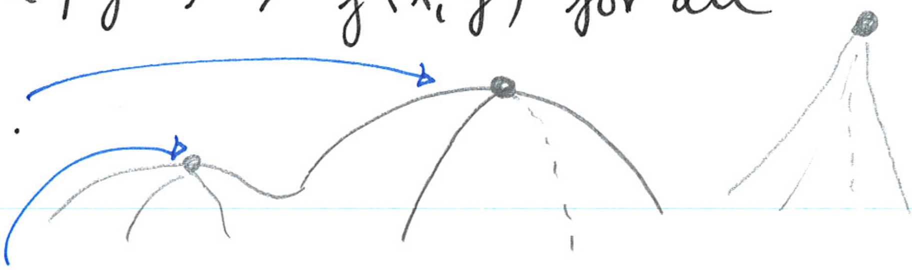
Def (Hessian): The Hessian of $f(x,y)$ is the 2×2 matrix

$$\mathcal{H}(f)(x,y) := \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

Optimization: max/min

Def (max/min):

i) (x^*, y^*) is a maximal point / maximizer for f if $f(x^*, y^*) \geq f(x,y)$ for all $(x,y) \in D_f$.

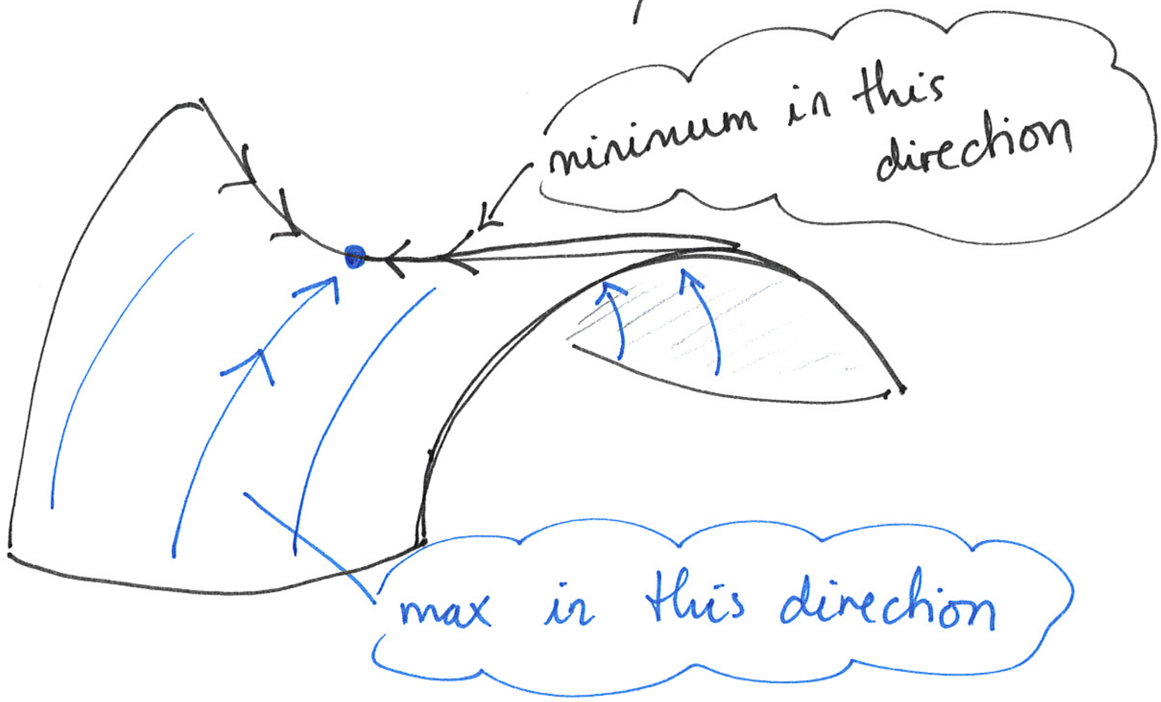


ii) (x^*, y^*) is a local max for f if $f(x^*, y^*) \geq f(x,y)$ for all (x,y) close to (x^*, y^*).

iii) (x^*, y^*) is a minimum point / minimizer for f if $f(x^*, y^*) \leq f(x,y)$ for all $(x,y) \in D_f$.

iv) (x^*, y^*) is a local minimum for f if $f(x^*, y^*) \leq f(x,y)$ for all (x,y) close to (x^*, y^*).

v) A stationary point (x^*, y^*) of f which is neither a local max. nor a local min. is called a <u>saddle point</u>.



<u>KEY RESULT</u>: If (x^*, y^*) is a max/min for f, then we have either:

i) (x^*, y^*) is a stationary point for f.
$$(f'_x = f'_y = 0 \text{ at } (x^*, y^*))$$

ii) Either f'_x or f'_y is not defined at (x^*, y^*).

iii) (x^*, y^*) is a boundary point of D_f.

The second derivative test

Result: If (x^*, y^*) is a stationary point of f, we compute

$$H(f)(x^*,y^*) = \begin{bmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{yx}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

We have that;

1) If $\underbrace{\det H(f)(x^*,y^*)}_{AC-B^2} > 0$ and $\underbrace{\operatorname{tr} H(f)(x^*,y^*)}_{\text{"trace": } A+C} > 0$, then (x^*, y^*) is a local min.

2) If $\det H(f)(x^*,y^*) > 0$ and $\operatorname{tr} H(f)(x^*,y^*) < 0$, then (x^*, y^*) is a local max.

3) If $\det H(f)(x^*,y^*) < 0$, then (x^*, y^*) is a saddle point.

NOTE: If $\det H(f)(x^*,y^*) = 0$, the test is inconclusive.

$\boxed{\text{D3-080}}$ ☺

$$f'_x(x,y) = \lim_{h \to 0} \frac{\overbrace{\left(3(x+h)+4y-5\right)}^{f(x+h,y)} - \overbrace{\left(3x+4y-5\right)}^{f(x,y)}}{h}$$

$$= \frac{\cancel{3x} + 3\cancel{h} + \cancel{4y} - \cancel{5} - \cancel{3x} - \cancel{4y} + \cancel{5}}{\cancel{h}}$$

$$= 3$$

Computation of the partial derivative via the definition: