Plan



# ① Computing inverse matrices

$A$ $n\times n$ matrix

$\underline{n=2}$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc} \cdot \overbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}^{\text{adj}(A)}$, $|A| \neq 0$

$|A| = ad - bc$ $\quad A^{-1}$ does not exist, $|A| = 0$

Note: $\boxed{\begin{array}{l} A \text{ is invertible } \Leftrightarrow A \text{ is square} \\ (A^{-1} \text{ exists}) \qquad\qquad \text{and } |A| \neq 0 \end{array}}$ holds for any $n$

Methods for computing $A^{-1}$ ($n$ any positive integer)

Method A: Adjoint matrix $\quad A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$

if $|A| \neq 0$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad |A| = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 6 - 5 + 1 = 2 \neq 0$

$A^{-1} = \frac{1}{2} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$

$C_{11} = +6 \quad C_{12} = -5 \quad C_{13} = +1$

$C_{21} = -6 \quad C_{22} = +8 \quad C_{23} = -2$

$C_{31} = +2 \quad C_{32} = -3 \quad C_{33} = +1$

$= \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$ : $\underbrace{\frac{1}{|A|} \cdot \text{adj}(A)}_{A^{-1}} \cdot A = I$ and $A \cdot A^{-1} = I$ $\Leftrightarrow$ $\text{adj}(A) \cdot A = |A| \cdot I$, $A \cdot \text{adj}(A) = |A| \cdot I$

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & \cdot & \cdot \\ C_{12} & \cdot & \cdot \\ C_{13} & \cdot & \cdot \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ - & - & \cdot \\ \cdot & - & \cdot \end{pmatrix}$$

$$A \cdot \operatorname{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ - & - & \cdot \\ - & - & \cdot \end{pmatrix} \cdot \begin{pmatrix} C_{11} & - & \cdot \\ C_{12} & - & \cdot \\ C_{13} & - & \cdot \end{pmatrix} = \begin{pmatrix} \overset{|A|}{\overset{\parallel}{a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}} & \cdot & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \cdot I$$

**Method B:** Gaussian process

$A$ : $n \times n$ $\quad (A \mid I) \to \cdots \to (B \mid C)$

elementary row operations — reduced echelon form

If $B = I$: $C = A^{-1}$
If $B \neq I$: $A^{-1}$ does not exist

Ex:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ : $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \rfloor -2 \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \cdot -1/3$

$\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \rceil -2 \to \left(\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$

$A^{-1} = \frac{1}{-3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

Theory:

An <u>elementary matrix</u> is any matrix you can get by doing <u>one</u> elementary row operation on the identity matrix.

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rfloor -2 \to \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = E_1$ elementary matrix

Fact: to multiply with $E_1$ from the left on any matrix $A$ is the same as using $\rfloor -2$ on $A$.

Ex: $A = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$ : $E_1 \cdot A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$

$$E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad -2$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1/3 \end{pmatrix} \quad 1 \cdot -1/3$$

$$E_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad -2$$

$$= \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}$$

$$\left(\begin{array}{cc} 3 & 5 \\ 4 & 6 \end{array}\right) -2 \rightarrow \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}$$

$$(A \mid I) \; -2 \rightarrow (E_1 A \mid E_1 \cdot I) \rightarrow (E_2 E_1 A \mid E_2 E_1 I)$$

$$\rightarrow (\underbrace{E_3 E_2 E_1 A}_{= I} \mid \underbrace{E_3 E_2 E_1 I}_{= E_3 E_2 E_1 = A^{-1}})$$

$$(E_3 E_2 E_1) \cdot A = I$$

$$A^{-1} = E_3 E_2 E_1$$

**Ex:** $n = 3$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right) \; -1 \; -1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right) \; -2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \; \cdot \tfrac{1}{2}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \; -3 \; -1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \; -1 \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

(Same as with method A)

**Fact:** If $|A| \neq 0$ and $AB = I$, then $BA = I$.
If $|A| \neq 0$ and $BA = I$, then $AB = I$.

② Problem 7, PS 35

a) $\underbrace{60}_{\text{price of A per sh.}} \underbrace{x}_{\text{\# shares of A}} + 75y + 320z = 400' = C$

Budget condition: Cost price of the portfolio is 400'.

b)
$$\begin{aligned} 20x + 5y + 30z &= R_1 \\ 40x - 50y + 180z &= R_2 \\ -20x + 25y - 265z &= R_3 \\ 60x + 75y + 320z &= C \end{aligned}$$

$$\left(\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right) \begin{array}{l} -2 \\ 1 \\ -3 \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 30 & -235 & R_3 + R_1 \\ 0 & 60 & 230 & C - 3R_1 \end{array}\right) \begin{array}{l} 1/2 \\ 1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|l} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2-2R_1 \\ 0 & 0 & -175 & R_3+R_1+\frac{1}{2}(R_2-2R_1) \\ 0 & 0 & 350 & C-3R_1+R_2-2R_1 \end{array}\right) \; 2$$

$$\rightarrow \left(\begin{array}{ccc|l} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2-2R_1 \\ 0 & 0 & -175 & R_3+\frac{1}{2}R_2 \\ 0 & 0 & 0 & C-3R_1+R_2-2R_1+2(R_3+\frac{1}{2}R_2) \end{array}\right)$$

$\cancel{C-4R_1+R_2+2R_3}$ $\quad C-5R_1+2R_2+2R_3$

Conclusion: Returns $(R_1,R_2,R_3)$ are possible $\iff$ $\cancel{C-4R_1+R_2+2R_3=0}$ $\cancel{\text{i.e. } 4R_1-R_2-2R_3=400'}$

$$C-5R_1+2R_2+2R_3=0$$

$$\boxed{5R_1-2R_2-2R_3=400'}$$

b) $R_1=50'$, $R_2=25'$, $R_3=-100'$

$5\cdot 50'-2\cdot 25'-2\cdot(-100')$
$=250'-50'+200'=400'$
Yes, it is possible

$$20x+5y+30z=50'$$
$$-60y+120z=-75'$$
$$-175z=-87{,}5'$$

$z=\frac{-87.5'}{-175}=1/2'=\underline{500}$

$-60y=-75'-120\cdot 0.5'=-135'$ $\quad y=\frac{135'}{60}=\underline{2250}$

$20x=50'-30\cdot 0.5'-5\cdot 2.25'=35'-11.25'$

$x=\frac{23.75'}{20}=\underline{1.187{,}5}$

Portfolio: $x=\underline{1187{,}5}$ $\quad y=\underline{2250}$ $\quad z=\underline{500}$

c) $R_1 > 0,\ R_2 = R_3 = 0$: $5R_1 = 400'$

$R_1 = 80' > 0$ Yes it is possible

Solve $20x + 5y + 30z = 80'$ $\quad z = 0$

$-60y + 120z = -160'$ $\quad -60y = -160'$

$-175z = 0$ $\quad y = \frac{160'}{60} = \frac{8'}{3} = 2\,^2/_3\,'$

$\cong 2.667$

$20x = 80' - 5\cdot(8/3)' = 80.000 - \frac{40.000}{3}$

$= \frac{3\cdot 80' - 40'}{3} = \frac{200'}{3} \Rightarrow x = \frac{200'}{3\cdot 20} = \frac{20'}{6} = \frac{10'}{3} \simeq 3.333$

Portfolio: $x = 3.333\,^1/_3$ $\quad y = 2.666\,^2/_3$ $\quad z = 0$

d) $R_1, R_2, R_3 > 0$: $5R_1 - 2(R_2 + R_3) = 400'$

Yes it is possible: For example $R_1 = R_2 = R_3$: $5R_1 - 4R_1 = R_1 = 400'$

$\Rightarrow R_1 = R_2 = R_3 = 400' > 0$

or $R_1 = 100' \Rightarrow 500' - 2(R_2 + R_3) = 400'$

$R_2 + R_3 = \frac{500' - 400'}{2} = 50'$

$R_2 = 25'$ $\quad R_3 = 25'$