# Examples

EBA1180
Lect. 36
Spring 24

**Ex:**

3×3 lin. syst.
$$\begin{cases} rx + 2y - z = 3 \\ x + (r+1)y - z = 3 \\ -x - 2y + rz = 1 - r \end{cases}$$

$x, y, z$: variables
$r$: parameter

$$|A| = \begin{vmatrix} r & 2 & -1 \\ 1 & r+1 & -1 \\ -1 & -2 & r \end{vmatrix}$$

$|A|$ ↓ coefficient matrix

Sign of cofactor expansion 3×3:
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= r \begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ -1 & r \end{vmatrix} - 1 \begin{vmatrix} 1 & r+1 \\ -1 & -2 \end{vmatrix}$$

$$= r\left((r+1)r - 2\right) - 2(r-1) - 1(-2 + r + 1)$$

$$= r(r^2 + r - 2) - 2r + 2 + 2 - r - 1$$

$$= r(r+2)(r-1) - 3r + 3$$

abc-formula etc. to factorize

$$= r(r+2)(r-1) - 3(r-1)$$

$$= (r-1)\left(r(r+2) - 3\right)$$

$$= (r-1)(r^2 + 2r - 3)$$

$$= (r-1)(r-1)(r+3) = \underline{(r-1)^2(r+3)}$$

2 ~~cases~~ cases :

1) $|A| = 0$ :   $(r-1)^2(r+3) = 0$

Either no solutions or $\infty$ many solutions → Which one ?

$\underline{r = 1}$ , $\underline{r = -3}$

i) $r = 1$ : Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

(-1) R1 add to R2
R1 add to R3

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Switch R2 and R3

Echelon form !

$0 = 3$ NOT TRUE!

⇒ No solutions for $r = 1$

ii) $r = -3$:

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right]$$

Switch R1 and R2 → To simplify calculations

3*R1 add R2
R1 add R3

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & -4 & -4 & 7 \end{array}\right]$$

(-1)*R2 add to R3

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

$\Rightarrow 0 = -5$ NOT TRUE!

$\Rightarrow$ No solutions for $r = -3$.

2) $|A| \neq 0$: $(r-1)^2 (r+3) \neq 0$

$r \neq 1$, $r \neq -3$ $\Rightarrow$ Unique solution

Find this via Cramer's rule: → Easier than Gaussian elimin. due to parameters

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To find x:

Know: $|A| = (r-1)^2 (r+3)$

Need $$|A_1(\vec{b})| = \begin{vmatrix} 3 & 2 & -1 \\ 3 & r+1 & -1 \\ 1-r & -2 & r \end{vmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \\ 1-r \end{bmatrix}$$

$$= 3 \begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 1-r & r \end{vmatrix}$$

$$+ (-1) \begin{vmatrix} 3 & r+1 \\ 1-r & -2 \end{vmatrix}$$

$$= \ldots = \underline{2r^2 - r - 1}$$

From Cramer's rule:

$$x = \frac{|A_1(\vec{b})|}{|A|} = \frac{2r^2 - r - 1}{(r-1)^2(r+3)}$$

1 is zero in numerator: abc-formula to factorize

$$= \frac{(2r+1)\cancel{(r-1)}}{(r-1)^{\cancel{2}}(r+3)}$$

$$= \underline{\underline{\frac{2r+1}{(r-1)(r+3)}}}, \quad r \neq 1, -3$$

Can find $y$ and $z$ similarly.

$$|A_2(\vec{b})| \qquad\qquad |A_3(\vec{b})|$$

## Vector equations

$$x \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}}_{\vec{v}_1} + y \underbrace{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{\vec{v}_2} + z \underbrace{\begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}}_{\vec{v}_3} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

A vector equation

$$\underbrace{x}_{c_1} \vec{v}_1 + \underbrace{y}_{c_2} \vec{v}_2 + \underbrace{z}_{c_3} \vec{v}_3 = \vec{b}$$

Can $\vec{b}$ be written as a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3$? How?

$$\begin{bmatrix} x \\ 0 \\ 4x \end{bmatrix} + \begin{bmatrix} 3y \\ -y \\ 2y \end{bmatrix} + \begin{bmatrix} 4z \\ -z \\ 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3y + 4z \\ -y - z \\ 4x + 2y + 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x + 3y + 4z &= 3 \\ -y - z &= 1 \\ 4x + 2y + 6z &= 2 \end{aligned}$$

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 4 & 2 & 6 & 2 \end{array}\right] \xrightarrow{-4} \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -10 & -10 & -10 \end{array}\right] \xrightarrow{-10}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -20 \end{array}\right] \Rightarrow \quad 0 = -20$$

Never true $\Rightarrow$

<u>No solutions!</u>

## Inverse matrices

**Def (Inverse matrices):** Let $A$ be an $\underbrace{n \times n}_{\text{square}}$ matrix.

An inverse of $A$ is a matrix $A^{-1}$ s.t.

$$A \cdot A^{-1} = I \quad \underline{\text{and}}$$

$$A^{-1} \cdot A = I$$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $|A| = 4 - 1 = 3 \neq 0$

$2^{-1} = \frac{1}{2}$

$2 \cdot 2^{-1} = 2 \cdot \frac{1}{2} = 1$

$2^{-1} \cdot 2 = \frac{1}{2} \cdot 2 = 1$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{some other matrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 4-1 & -2+2 \\ 2-2 & -1+4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Can check (DIY):

$$\underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{the other matrix}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Hence (from def.): $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

**FORMULA (inverse, $n=2$):**

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - bc \neq 0$:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|A| = 0 = ad - bc$:

No inverse of $A$.

FACTS: i) The inverse of $A$ does not always exist. Actually, $A$ is invertible (i.e. $A^{-1}$ exists) if and only if $|A| \neq 0$.

ii) If $A$ has an inverse, it is unique.

iii) General formulla for $A^{-1}$ when $|A| \neq 0$:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

where $C_{ij}$ are the cofactors.

oo

Ex:

$$2x + y = 4$$
$$x + 2y = 3$$

Matrix form:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{b}}$$

If $A^{-1}$ exists:

$$\underbrace{A^{-1}A}_{I}\,\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$2x = b$$

$$\underbrace{2^{-1}\cdot 2}_{1}x = 2^{-1}b$$

$$x = 2^{-1}b$$

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$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

$$\vec{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8-3 \\ -4+6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

So: $\underline{\underline{(x,y) = \left(\frac{5}{3}, \frac{2}{3}\right)}}$

Advantage of $A^{-1}$: Quickly solve lin. syst. for many diff. r.h.s.