Linear combinations

EBA 1180
Spring 24
Lect. 35

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$

three 3-vectors.

Def: (Linear combination)

A linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an expression of the form:

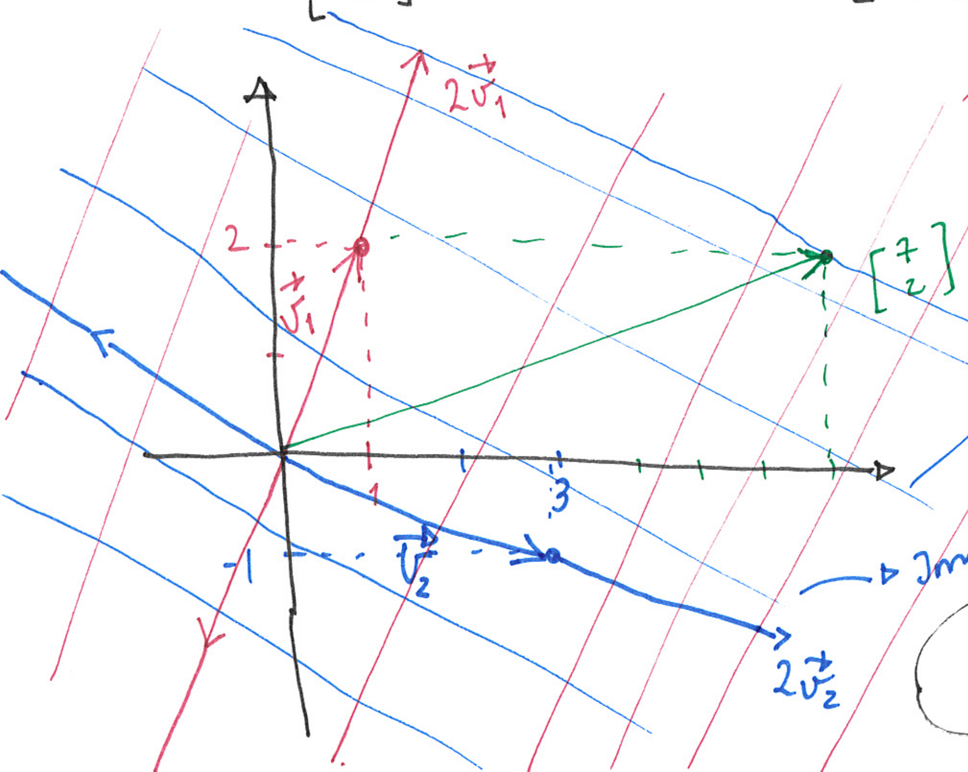
$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

where c_1, c_2, c_3 are given numbers.

In gen: $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$ where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are m-vectors

Ex: Is $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ a linear combination of

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$?



Scalar multiples of $\vec{v}_1$

Scalar multiples of $\vec{v}_2$.

Imagine this is parallel

$\vec{v}_1$ and $\vec{v}_2$ span the whole plane

Want to find c_1 and c_2 s.t:

$$\underbrace{c_1 \vec{v}_1 + c_2 \vec{v}_2}_{\text{linear combination of } \vec{v}_1, \vec{v}_2} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \rightarrow \text{A vector equation}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{array}{l} x + 3y = 7 \\ 2x - y = 2 \end{array} \leftarrow \left.\begin{array}{l} c_1 + 3c_2 = 7 \\ 2c_1 - c_2 = 2 \end{array}\right\} 2 \times 2 \text{ lin. syst.}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -12 \end{array}\right]$$

$(-2) *$ row 1
add to row 2

$$\begin{array}{l} c_1 + 3c_2 = 7 \\ \quad\; -7c_2 = -12 \end{array} \Rightarrow c_2 = \frac{12}{7} \approx \underline{1,72}$$

$$c_1 = 7 - 3c_2 = \ldots = \frac{13}{7} \approx \underline{1,86}$$

$$(c_1, c_2) \approx (1,86, 1.72)$$

$$1,86 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1.72 \begin{bmatrix} 3 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\frac{13}{7} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{12}{7} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Matrix multiplication

Def: (Matrix multiplication)

A, B matrices and # columns in A = # rows in B.

Then,

$$\underbrace{A \cdot B}_{\text{matrix multiplication of } A \text{ and } B} \text{ is defined.}$$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 4 \\ 2\cdot 3 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$$A \qquad \cdot \qquad B$$

$$\underset{\text{\# columns in } A}{2 \times 2} \quad \cdot \quad \underset{\text{\# rows in } B}{2 \times 1} = 2 \times 1$$

matrix multipl. is defined

Why defined like this?

EQN. FORM: $\begin{cases} x + 2y = 5 \\ 2x + y = 1 \end{cases}$ $\qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A \cdot \vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

$2 \times 2 \cdot 2 \times 1 = 2 \times 1$

SAME!

MATRIX FORM: $\{ A \cdot \vec{x} = \vec{b} \rightarrow \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Can take powers of square matrices:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$2 \times 2 \cdot 2 \times 2 \rightsquigarrow 2 \times 2$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 & 1\cdot 2 + 2\cdot 1 \\ 2\cdot 1 + 1\cdot 2 & 2\cdot 2 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Formula for $A \cdot B$: If $A \cdot B$ is defined with $A = [a_{ij}]$, $B = [b_{ij}]$, then $A \cdot B = C = [c_{ij}]$ where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

NOTE: $AB \neq BA$

Even though AB is defined, BA may not be.

Ex: $\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

$2 \times 2 \cdot 2 \times 1 = 2 \times 1 \Rightarrow AB$ defined.

$$\underset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}$$

$2 \times 1 \quad \cdot \quad 2 \cdot 2$ $\Rightarrow$ BA not defined.

NOT SAME: $1 \neq 2$

Linear systems

Ex:

$$\left.\begin{aligned} x + y + z + w &= 4 \\ x - y \qquad + 2w &= 7 \\ 2x + 3y - z \qquad &= 10 \end{aligned}\right\}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + y + z + w \\ x - y \quad + 2w \\ 2x + 3y - z \end{bmatrix}$$

(Dim: $3 \times 4 \cdot 4 \times 1 = 3 \times 1$)

$$= \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} = \vec{b}$$

So the lin. syst. can be written:

$$\boxed{A\vec{x} = \vec{b}}$$

The matrix form of the linear system

Matrix algebra

1) Addition, subtraction: $A+B$, $A-B$ (A, B same size)

2) Scalar multiplication: $c \cdot A$, c is a number (always defined)

3) Matrix multiplication: $A \cdot B$ (# columns in A = # rows in B)

4) Powers: A^n (defined for A square)

Ex: $\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{2\times 2,\ \text{square}}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}_{2\times 3,\ \text{not square}}^2 = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}_{2\times 3} \cdot \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}_{2\times 3}$

$\Rightarrow$ Not defined

Special matrices

$\rightarrow$ The identity matrix:

$I_2 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2}$, $I_3 = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{3\times 3}$, ...

$I_n \rightarrow$ identity matrix of size n

→ We say that: $A^0 = I$

Property:

- $A \cdot I = A$
  Ex: $(2\times 3 \cdot 3\times 3)$
- $I \cdot A = A$
  Ex: $(2\times 2 \cdot 2\times 3)$

for any A, I of suitable dimension

Ex:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot 0 + 2\cdot 1 \\ 2\cdot 1 + 1\cdot 0 & 2\cdot 0 + 1\cdot 1 \end{bmatrix}$$

$$2\times 2 \cdot 2\times 2$$

$$= \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1\cdot 1 + 0\cdot 2 & 1\cdot 2 + 0\cdot 1 \\ 0\cdot 1 + 1\cdot 2 & 0\cdot 2 + 1\cdot 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}$$

The transpose

$$\underset{m\times n \text{ matrix}}{A} \xrightarrow{\text{transpose}} \underset{n\times m \text{ matrix}}{A^T}$$

A^T → "A transpose" OR "transpose of A"

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}_{2\times 3}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}_{3\times 2}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 7 & 3 \end{bmatrix}_{3\times 3}, \quad A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 7 \\ 0 & 4 & 3 \end{bmatrix}_{3\times 3}$$

Main diagonal is preserved when we take the transpose of a square matrix

Def: A is a symmetric matrix if $A = A^T$.

Ex:

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix} = B$$

$B = B^T$, hence B is symmetric.

Rules for matrix algebra:

"Normal:"

- $A + B = B + A$
- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $A \cdot (BC) = (AB) \cdot C$

"NOT normal:"

- $AB \neq BA$
- $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

$$\neq A^2 + 2AB + B^2$$

Determinants

- $|A \cdot B| = |A| \cdot |B|$
- $|cA| = c^n |A|$, for A $n \times n$
- $|A^T| = |A|$

All can be proved, but none shall

Transpose:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

$$\boxed{C2\text{-}010}$$