Determinants and linear systems

EBA1180
Sect. 34
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$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

Cofactor expansion:

$$- 0 \begin{vmatrix} \dots \end{vmatrix}$$

$$+ 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix}$$

$$= -0 \begin{vmatrix} \dots \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 0 \begin{vmatrix} \dots \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix}$$

$$= -1 \left(1 \cdot (-1) - 1 \cdot 1\right) - 1 \left(1 \cdot (-1) - 1 \cdot 1\right)$$

$$= -1 (\overbrace{-1-1}^{-2}) - 1 (\overbrace{-1-1}^{-2})$$

$$= 2 + 2 = \underline{\underline{4}}$$

Alternative method for finding the determinant

1) Gaussian elimination until upper triangular matrix → NB: May be scaling and/or change of signs

2) Determinant is then the product of the diagonal elements.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \underset{\substack{(-1)*\text{row } 1 \\ \text{add to row } 3}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\underset{\substack{(-1)*\text{row } 2 \\ \text{add to row } 4}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = E$$

RESULT: If E is an upper triangular matrix (i.e. all entries below main diagonal are 0), then $|E|$ is the product of diagonal entries.

NOTE: All echelon forms are upper triangular.

Ex. ctd.:

i) $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2) = \underline{\underline{4}}$

ii) NB: Here $|A| = |E| = 4$

Why is the result true?

Ex:

Cofactor expansion

$$|E| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$- 0 \begin{vmatrix} \ddots \end{vmatrix}$$

$$+ 0 \begin{vmatrix} \ddots \end{vmatrix} - 0 \begin{vmatrix} \ddots \end{vmatrix}$$

$$= 1 \cdot 1 \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 1 \cdot 1 \cdot \big((-2) \cdot (-2) - 0 \big)$$

$$= 1 \cdot 1 \cdot (-2) \cdot (-2) = 4$$

Pattern holds for any upper triangular matrix.

Ex: $\begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 2 \cdot 0 = \underline{0}$

Q: How many solutions of corresp. lin. syst.?

0 or ∞ many!

RESULT (Change in determinant from elementary row operations)

If $A \sim B$ via elementary row operations, then

i) Switch two rows $\Rightarrow |B| = -|A|$

ii) Multiply a row by $c \neq 0 \Rightarrow |B| = c|A|$

iii) Add a multiple of a row to another row:

$$|B| = |A|$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $|A| = 0 \cdot 1 - 1 \cdot 1 = \underline{-1}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$$

switch row 1 and 2

$$|B| = 1 \cdot 1 - 1 \cdot 0 = \underline{1}$$
$$= -|A|$$

Recall:

Result: i) $|A| \neq 0 \Rightarrow$ One solution

ii) $|A| = 0 \Rightarrow$ no solutions <u>or</u> ∞ many solutions

Ex:

$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}\right\} (*)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$$= 2 \neq 0, \text{ so } |A| \neq 0 \Rightarrow$$

$(*)$ has one unique solution (from Result)

Check via row reduction:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix}$$

(−1) * row 1 add to row 2, (−1) * row 1 add to row 3

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

(−2) * row 2 add row 3

Pivot in each diagonal element: So unique solution

Ex. alt:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ 2x + 3y + 5z &= 10 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 10 - 12 - (5-8) + 3 - 4$$

$$= -2 + 3 - 1 = 0 \Rightarrow$$ no solutions OR infinitely many solutions

Result

Which one?

Check via row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 5 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array}\right]$$

(−1)* row 1 add to row 2, (−2)* row 1 add to row 3

(−1)* row 2 add to row 3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$ ∞ many solutions

Linear systems with parameters

Ex:
$$x + y = 4$$
$$x + ay = 6$$

x, y : variables → endogeneous

a : parameter → solution depends on the parameter

exogeneous

Solve: Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

(-1)* row 1
add to row 2

Pivot or not?

Depends!

Two cases:

$a - 1 = 0$

$\boxed{a = 1}$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right]$$

$\Downarrow$

No solutions!

$a \neq 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

Pivot!

$$x + y = 4$$
$$(a-1)y = 2 \Rightarrow y = \frac{2}{a-1}$$

$$x = 4 - y = 4 - \frac{2}{a-1}$$
$$= \frac{4a - 4 - 2}{a-1}$$
$$= \frac{4a-6}{a-1}$$

Solution:

$$(x, y) = \left(\frac{4a-6}{a-1}, \frac{2}{a-1} \right), \quad a \neq 1$$