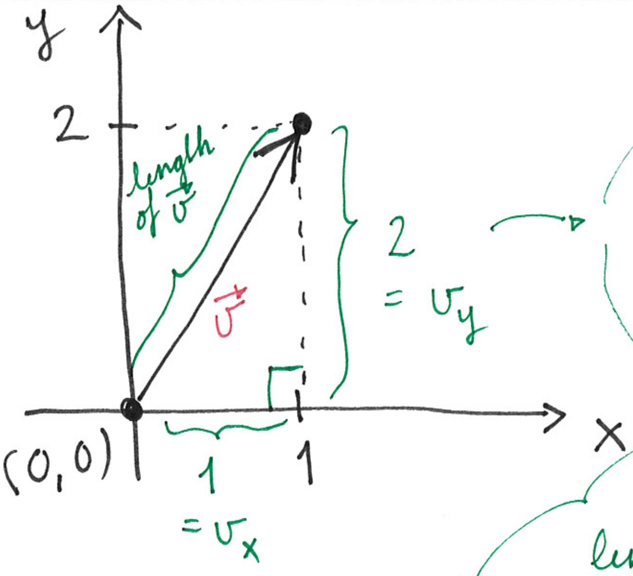
Geometric interpretation of vectors

EBA1180
Sect. 33
Spring 24

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

corresp. to an arrow from $(0,0)$ to (v_x, v_y)
$(1, 2)$



Pythagorean theorem:

$$v_x^2 + v_y^2 = \left(\text{length of } \vec{v}\right)^2$$

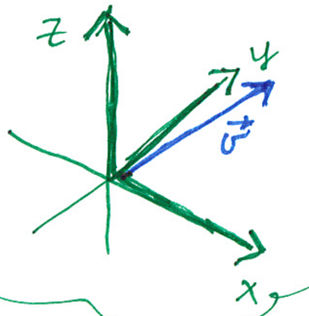
$$\text{length of } \vec{v} = \sqrt{v_x^2 + v_y^2}$$

- A vector has length (magnitude) and a direction.

$$\|\vec{v}\| = \left\| \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right\| = \sqrt{v_x^2 + v_y^2}$$

Ex: $\|\vec{v}\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$

3-dimensions:



Def: (Length of vector)

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

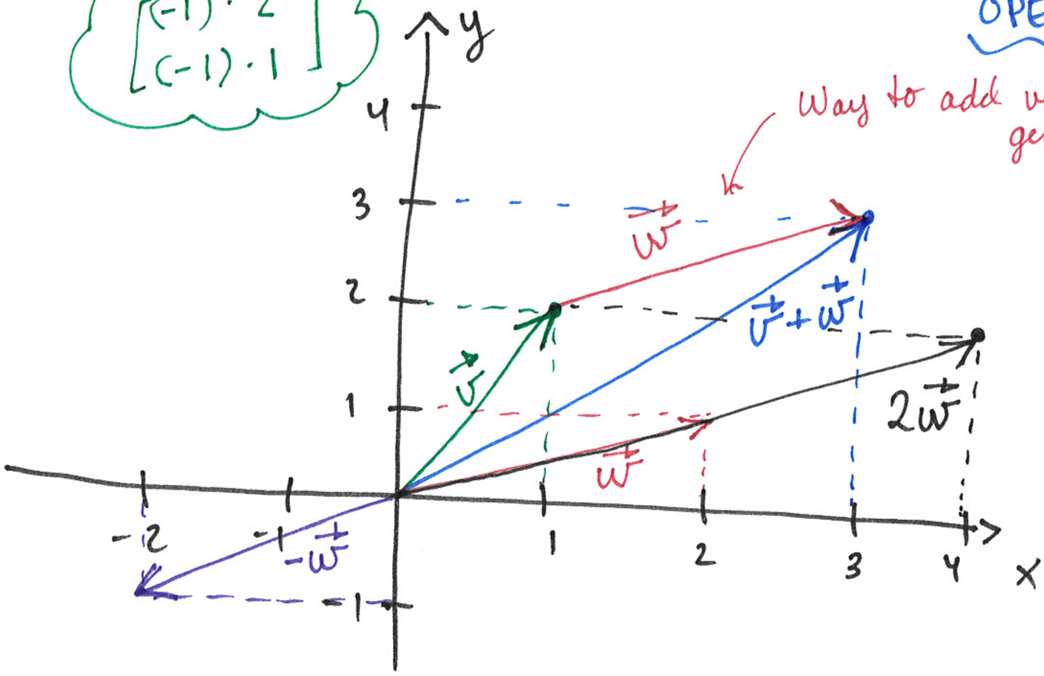
ADD: $\vec{v} + \vec{w} = \begin{bmatrix} 1+2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

SCALAR MULTIPLICATION: $2\vec{w} = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$(-1)\vec{w} = -\vec{w} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} (-1) \cdot 2 \\ (-1) \cdot 1 \end{bmatrix}$

VISUALIZE VECTOR OPERATIONS:



Way to add vectors geometrically

Determinants

"the determinant of A"

$$\underset{\substack{n \times n \\ \text{matrix}}}{A} \longrightarrow \overbrace{\det}(A) = \overbrace{|A|}$$

a **number**

(square !)

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

2×2

$$= 2 \cdot 2 - 1 \cdot 1$$

$$= \underline{\underline{3}}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

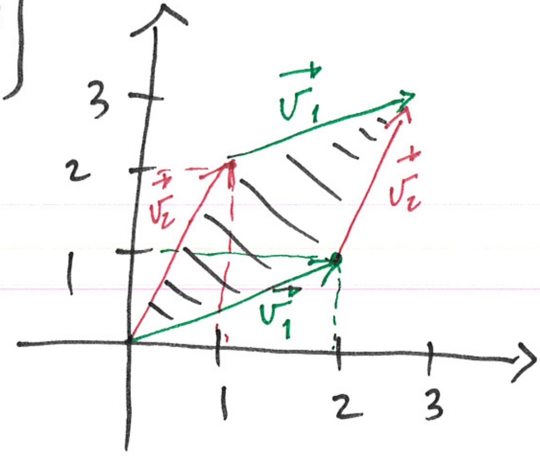
$$= ad - bc$$

FORMULA: (Determinant, $n=2$)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Interpretation:

$$A = \begin{array}{cc} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \end{array}$$



$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \underline{3}$$

Can prove: $|\det(A)|$ = area of parallellogram spanned by $\vec{v}_1$ and $\vec{v}_2$.

NOTE:

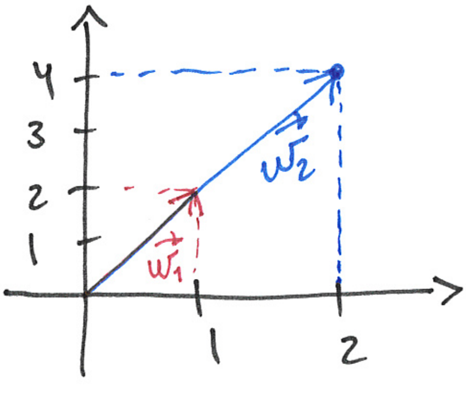
$$\left| \vec{v}_2 \ \vec{v}_1 \right| = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot 2 = \underline{-3}$$

Area can 4 be neg: Need abs. value

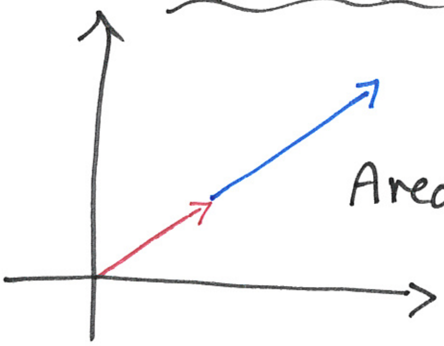
Ex:

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0$$

$$\vec{w}_1 \quad \vec{w}_2$$



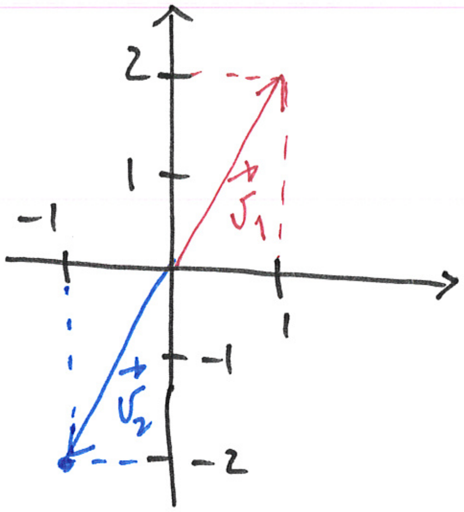
Parallellogram spanned by $\vec{w}_1$ & $\vec{w}_2$:



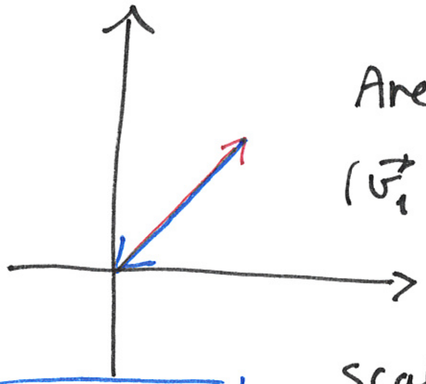
($\vec{v}_1$ and $\vec{v}_2$ are scalar multiples)

Ex: $$\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 1\cdot(-2) - (-1)\cdot 2 = 0$$

$\vec{v}_1$ $\vec{v}_2$



Parallellogram spanned by $\vec{v}_1$ and $\vec{v}_2$:



Area = 0

($\vec{v}_1$ and $\vec{v}_2$ are scalar multiples)

Result: (Determinant = 0, n = 2)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

satisfies: One of the vectors is a scalar multiple of the other

The general case, n × n : METHOD for finding determinants

Cofactor expansion

A, n × n matrix.

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ 3×3 ; square

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$$

cofactor expansion along row 1

where C_{11}, C_{12}, C_{13} are **cofactors**:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

where M_{ij} is the determinant of the submatrix you get when you delete row i, column j.

minor

So:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1 \cdot (-1) \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix}$$

$$+ 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

NOTE: → Cofactor expansion along any row / column gives the same result.

→ Any determinant can be computed by cofactor expansion.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Ex:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ -1 & 1 & -1 \end{vmatrix}$$

$$- 1 \cdot \begin{vmatrix} 1 & 4 & 8 \\ 1 & 9 & 27 \\ 1 & 1 & -1 \end{vmatrix}$$

$$+ 1 \cdot \begin{vmatrix} 1 & 2 & 8 \\ 1 & 3 & 27 \\ 1 & -1 & -1 \end{vmatrix}$$

$$- 1 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & -1 & 1 \end{vmatrix} = \ldots$$

→ $\boxed{\text{Exploit the 0's !}}$ Use row/column with lots of 0's to simplify computations.

Connection between the # solutions of linear systems and determinants

- If you start with n×n linear system (# eqns = # variables), then the corresp. **coefficient matrix** is an n×n matrix.

Ex:

$$\underbrace{\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}}_{3\times 3 \text{ linear system}}$$

Write this as: $A\vec{x} = \vec{b}$ (matrix form)

where

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{\text{Coefficient matrix; } 3\times 3 \text{ matrix}}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}$$

→ Since coefficient matrix of n×n lin. syst. is an n×n matrix, we can $|A|$.

Result: i) $|A| \neq 0$ ⇒ One unique solution

not equal to

ii) $|A| = 0$ ⇒ No solutions or infinitely many solutions.

Theorem (Cramer's rule):

Consider a linear system, $A\vec{x} = \vec{b}$, with coefficient matrix A and r.h.s. $\vec{b}$, s.t. A is square $(n\times n)$ with $|A| \neq 0$. Then, the solution of the linear system is:

$$x_1 = \frac{|A_1(\vec{b})|}{|A|}, \quad x_2 = \frac{|A_2(\vec{b})|}{|A|}, \ldots,$$

$$x_n = \frac{|A_n(\vec{b})|}{|A|}$$

where $A_i(\vec{b})$ is the matrix you get when you replace the i'th column in A with $\vec{b}$.

o—o

Ex: $x + y = 4$, x, y : variables

$x + ay = 6$ a : parameter