More linear systems:
Number of solutions

6/2-24
EBA 1180
Sect. 32

Ex:

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\
x_1 - 2x_2 - x_3 + 4x_5 &= 8 \\
2x_1 + x_2 - 5x_3 + 7x_4 &= 11
\end{aligned}
$$

Def (Pivot position):
A pivot position is a position where there is a pivot in the echelon form.

Ex ctd:

$$
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 17 \\
1 & -2 & -1 & 0 & 4 & 8 \\
2 & 1 & -5 & 7 & 0 & 11
\end{array}\right]
$$

Add $(-1)*$ row 1 to row 2
Add $(-2)*$ row 1 to row 3

$$
\sim \left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 17 \\
0 & -3 & -2 & -1 & 3 & -9 \\
0 & -1 & -7 & 5 & -2 & -23
\end{array}\right]
$$

switch rows 2 & 3

$$
\sim \left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 17 \\
0 & -1 & -7 & 5 & -2 & -23 \\
0 & -3 & -2 & -1 & 3 & -9
\end{array}\right]
$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & 0 & 19 & -16 & 9 & 60 \end{array}\right]$$

add $(-3)*$ row 2 to row 3

Echelon form!

Pivots! Pivot positions are

$(1,1)$, $(2,2)$, $(3,3)$.

The linear system has two degrees of freedom (x_4, x_5 are free). Hence, it has infinitely many solutions.

Why? From echelon form

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ -x_2 - 7x_3 + 5x_4 - 2x_5 &= -23 \\ 19x_3 - 16x_4 + 9x_5 &= 60 \Rightarrow \end{aligned}$$

$$19x_3 = 60 + 16x_4 - 9x_5$$

$$\vdots$$

$x_2 = \ldots$ via x_4 and x_5 ...

$x_1 = \ldots$ via x_4 and x_5 ...

$\rightsquigarrow$ Can choose any x_4 and x_5 and the original linear system still holds.

Result: For any linear system, the pivot positions determine the number of solutions.

Different cases:

i) Pivot position in the last column:

No solutions.

Ex:

$$\left[\begin{array}{ccc|c} \vdots & \cdots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right] \longrightarrow 0 = 1 \text{ ; Never true}$$

Says:

ii) No pivot position in the last column:

The linear system has solutions.

a) Pivot positions in all variable columns:

One solution

Ex:

$$\left[\begin{array}{ccc|c} 1 & - & - & \vdots \\ 0 & 1 & - & \vdots \\ 0 & 0 & 1 & \vdots \end{array}\right] \begin{array}{l} \longrightarrow x_2 = \cdots \\ \longrightarrow x_3 = \text{number} \end{array}$$

b) There are variable columns without pivot positions: Infinitely many solutions

Ex:

$$\left[\begin{array}{cccc|c} 1 & \cdots & - & - & \vdots \\ 0 & 1 & \cdot & - & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \end{array}\right]$$

Theorem: Any linear system has either:

i) No solutions ⇝ Inconsistent

ii) One unique solution

iii) Infinitely many solutions.

} ⇝ Consistent

Computations with matrices and vectors

Def (m × n matrix):

An m × n matrix is a rectangular array of numbers with m rows and n columns.

Ex:

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{bmatrix}}_{3 \text{ columns}} \Big\} \; 2 \text{ rows}$$

Capital letter for matrices

Dimension: 2 × 3 matrix

Read: ("2 by 3" or "2 times 3")

$$\begin{array}{cccc} & & & \\ A = & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \\ & \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & \end{array}$$

a_{12}

row 1

column 2

- **Addition**: $A + B$
- **Subtraction**: $A - B$

Defined if A and B have the same size (e.g. both $m \times n$)

Result is a matrix of the same size as A/B

- **Scalar multiplication**: $\underbrace{r \cdot A}_{\text{Always defined}}$

Result is a matrix of same size as A

r : scalar (number)

A : matrix

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+(-1) & 3+1 \\ -1+1 & 0+2 & 2+3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

Do addition/subtraction position by position.

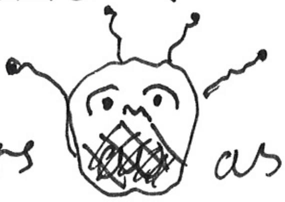
Ex:

$$2 \cdot \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 4 \\ 2\cdot(-1) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -2 & 4 \\ 0 & 2 \end{bmatrix}$$

Do multiplication by scalar position by position.

Def (n-vector)

An n-vector is a matrix with n rows and 1 column (a column vector).

• Write vectors  as $\vec{v}$ = boldface $\mathbf{v}$ = $\underline{v}$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; a 3-vector viewed as a column vector

Vector operations

lower case letters for vectors

→ Addition: $\vec{v} + \vec{w}$

→ Subtraction: $\vec{v} - \vec{w}$

→ Scalar multiplication: $r \cdot \vec{v}$ (r scalar) — number

Ex:

ADD: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}}$

SUBTRACT: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2-(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}}$

SCALAR MULTIPLICATION:

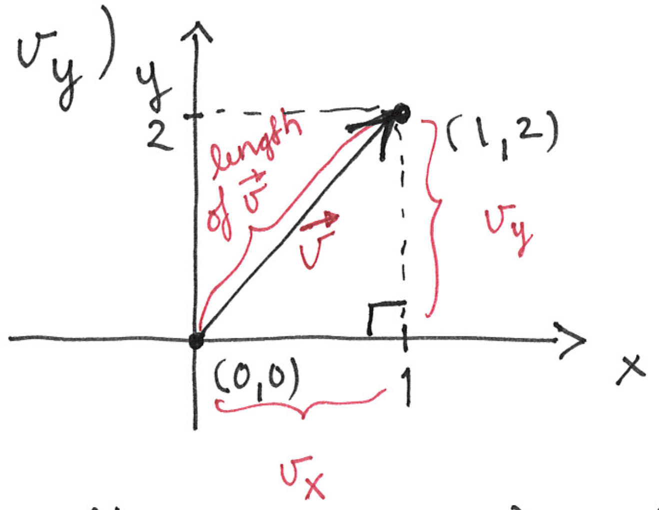
$$2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$(-1) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Geometric interpretation of vectors

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

corresponds to an arrow from $(0, 0)$ to (v_x, v_y)



A vector has a length (magnitude) and a direction.