New theme:

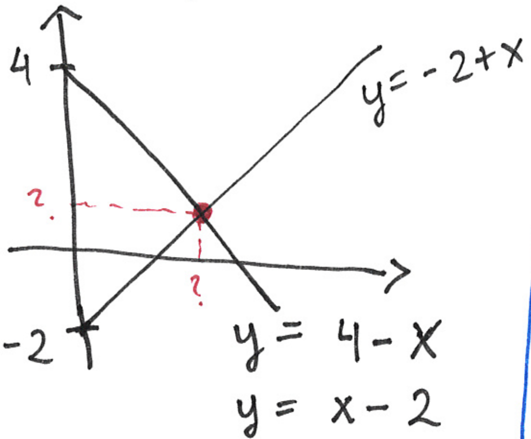
Systems of equations

EBA 1180
Lect. 31
Spring 24

Some types of systems of equations:

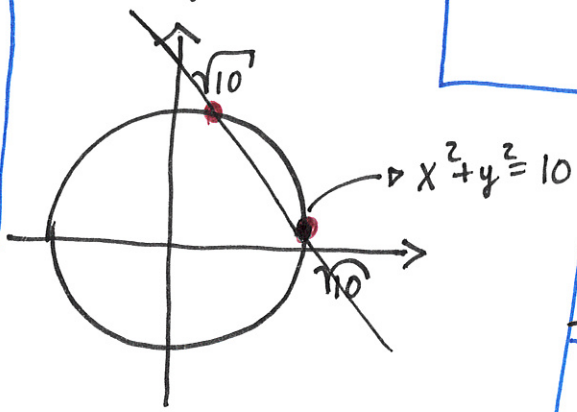
LINEAR:

i) $x+y=4$
$x-y=2$

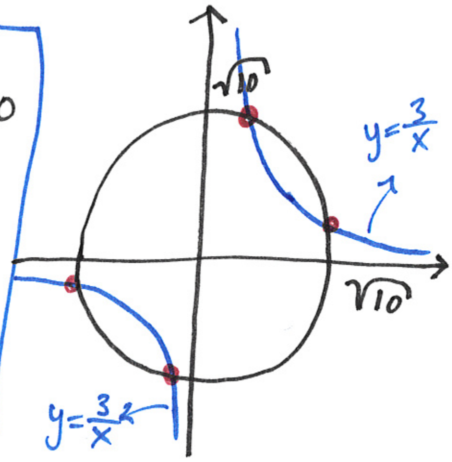


NON-LINEAR:

ii) $x^2+y^2=10$
$x+y=4$



iii) $x^2+y^2=10$
$xy=3$



$xy=3 \Rightarrow y=\frac{3}{x}$

SOLVE?

i) **2 methods**

Eliminate:

$$\begin{array}{rl} & x+y=4 \\ + & x-y=2 \\ \hline & 2x=6 \\ & \underline{x=3} \end{array}$$

$y=3-2=\underline{1}$

$(x,y)=(3,1)$

Substitute:

$x+y=4 \Rightarrow y=4-x$

$x-y=2$

$x-(4-x)=2$

$2x-4=2$

$2x=6$

$\underline{x=3}$

$y=4-3=\underline{1}$

SAME!

$(x,y)=(3,1)$

ii) $x+y=4$

$y=4-x$

$x^2+(4-x)^2=10$

$x^2+16-8x+x^2=10$

$2x^2-8x+6=0$

$x^2-4x+3=0$

$$x=\frac{4\pm\sqrt{16-4\cdot1\cdot3}}{2\cdot1}$$

$$=\frac{4\pm2}{2} \Rightarrow \underline{x_1=3},\ \underline{x_2=1}$$

$y_1=4-x_1=4-3=\underline{1}$ and $y_2=4-x_2=4-1=\underline{3}$

$\Rightarrow$

$(x_1,y_1)=(3,1)$ and $(x_2,y_2)=(1,3)$

iii) $xy = 3$

$$y = \frac{3}{x}$$

$$x^2 + \left(\frac{3}{x}\right)^2 = 10$$

$$x^2 + \frac{9}{x^2} = 10$$

$$x^4 + 9 = 10x^2$$

$$\boxed{x^2 = u}$$

$$x^4 - 10x^2 + 9 = 0$$

$$\boxed{\text{TRICK}}$$

$$\underbrace{(x^2)^2}_{u^2} - 10\underbrace{(x^2)}_{u} + 9 = 0$$

$$u^2 - 10u + 9 = 0 \longrightarrow u = \frac{10 \pm \sqrt{(-10)^2 - 4\cdot 1\cdot 9}}{2\cdot 1}, \quad u_1 = 9, \; u_2 = 1$$

abc-formula $\Rightarrow x_1^2 = 9, \; x_2^2 = 1$

$$x_1 = \pm 3, \quad x_2 = \pm 1 \qquad \boxed{y = \frac{3}{x}}$$

$$\Downarrow$$

$$\{(x, y)\} = \{(3, 1), (-3, -1), (1, 3), (-1, -3)\}$$

Def: (Linear system)

An $m \times n$ linear system in the variables $x_1, x_2, \dots, x_n$ is a system of m linear equations in $x_1, \dots, x_n$. It has the form:

$$\left.\begin{array}{l} \overbrace{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n}^{n \text{ unknowns/variables}} = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\} m \text{ eqns.}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are given numbers.

Ex:

$$\text{3 eqns.}\left\{\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 7 \\ x_1 - x_2 + 0 \cdot x_3 + 2x_4 = 10 \\ x_1 + x_2 - x_3 = 3 \end{array}\right.$$

4 variables

$\leadsto$ 3×4 linear system.

Ex:

$$\text{3 eqns}\left\{\begin{array}{ll} x + y + z = 3 & (1) \\ x + 2y + 4z = 7 & (2) \\ x + 3y + 9z = 13 & (3) \end{array}\right. \leadsto 3 \times 3 \text{ linear system.}$$

3 variables

SOLVE?

(1) $x = 3 - y - z$

(2) $3 - y - z + 2y + 4z = 7$

$\boxed{y + 3z = 4} \Rightarrow y = 4 - 3z$

(3) $3 - y - z + 3y + 9z = 13$

$\boxed{2y + 8z = 10}$

$\Rightarrow 2(4 - 3z) + 8z = 10$

$2z = 2$

$\underline{z = 1}$

$y = 4 - 3 \cdot 1 = \underline{1}$

$x = 3 - 1 - 1 = \underline{1}$

Solution: $(x, y, z) = (1, 1, 1)$

Gaussian elimination

General and systematic method to solve any linear system.

METHOD

t.b.e

1) Write down the augmented matrix of the system.

2) Use elementary row operations until you reach echelon form

3) Use back substitution to solve the system.

Ex:

$$x + y + z = 3$$
$$x + 2y + 4z = 7$$
$$x + 3y + 9z = 13$$

1) Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right]$$

Elementary row operations:

i) switch two rows

ii) Multiply a row by a constant $c \neq 0$.

iii) Add a multiple (by a constant (non-zero)) of a row to another row.

2) Gaussian:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right]$$

row equivalent to

-1 times first row and add to second row

$[-1 \ -1 \ -1 \mid -3]$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \end{array} \quad \begin{aligned} y + 3z &= 4 \\ 2y + 8z &= 10 \end{aligned}$$

-1 times row 1, add to row 3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \quad \begin{aligned} y + 3z &= 4 \\ 2z &= 2 \end{aligned}$$

(−2) * row 2 add to row 3

$[0 \ -2 \ -6 \mid -8]$

Echelon form!
Trappeform

Pivot: The first non-zero element in a row is called a pivot.

Echelon form: An echelon form is where

1) All entries below a pivot are 0.
2) If some rows are all zeros, they're at the bottom of the matrix.

3) Gaussian: To solve from echelon form:

Back substitution:

1) Start with last equation.
2) Work backwards & substitute the variables we've solved for.

Ex:

$$\begin{aligned} x + y + z &= 3 &\Rightarrow\ & x + 1 + 1 = 3 \Rightarrow \underline{x = 1} \\ y + 3z &= 4 &\Rightarrow\ & y + 3 \cdot 1 = 4 \Rightarrow \underline{y = 1} \\ 2z &= 2 &\Rightarrow\ & \underline{z = 1} \end{aligned}$$

Solution:

$$(x, y, z) = (1, 1, 1)$$

How to determine the number of Solutions

Ex:

$$x + y + z = 4$$
$$x - y + z = 2$$
$$x + 5y + z = 8$$

(-1) * row 1, add to row 2 and 3

$$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 1 & -1 & 1 & | & 2 \\ 1 & 5 & 1 & | & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & -2 & 0 & | & -2 \\ 0 & 4 & 0 & | & 4 \end{bmatrix}$$

x-column, y-column, z-column

2 * row 2, add to row 3

$$\sim \begin{array}{ccc} x\text{-column} & y\text{-column} & z\text{-column} \\ \end{array} \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & -2 & 0 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

NO PIVOT → Free variable

Echelon form!

⇝ • Pivot: Basic variable: x, y

• Vertical column with no pivot: Free variable: z

$$x + y + z = 4 \Rightarrow x + 1 + z = 4 \Rightarrow \underline{x = 3 - z}$$
$$-2y = -2 \Rightarrow \underline{y = 1}$$

Solution: $(x, y, z) = (3 - z, 1, z)$

where z is free.

∞ many solutions

One more case:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} x + y + z &= 4 \\ -2y \quad &= -2 \\ 0 &= 1 \longrightarrow \text{NEVER TRUE!} \end{aligned}$$

$\Downarrow$

No solutions

General: Pivot in last column of an (extended) echelon form $\Longleftrightarrow$ no solutions.

Now: CIU-067 (?) Check calendar!