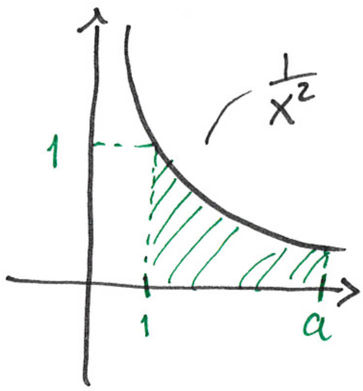
EBA 1180
Sect. 6 (30)
Spring 24

Recap: Improper integrals

Exercise: $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

$$= \lim_{a\to\infty} \int_1^{a} \frac{1}{x^2}\,dx$$



see notes last lecture

$$= \lim_{a\to\infty} \left[-\frac{1}{x}\right]_{x=1}^{a}$$

$$= \lim_{a\to\infty}\left(-\frac{1}{a} - \left(-\frac{1}{1}\right)\right) = \lim_{a\to\infty} -\frac{1}{a} + 1$$

$$= 1$$

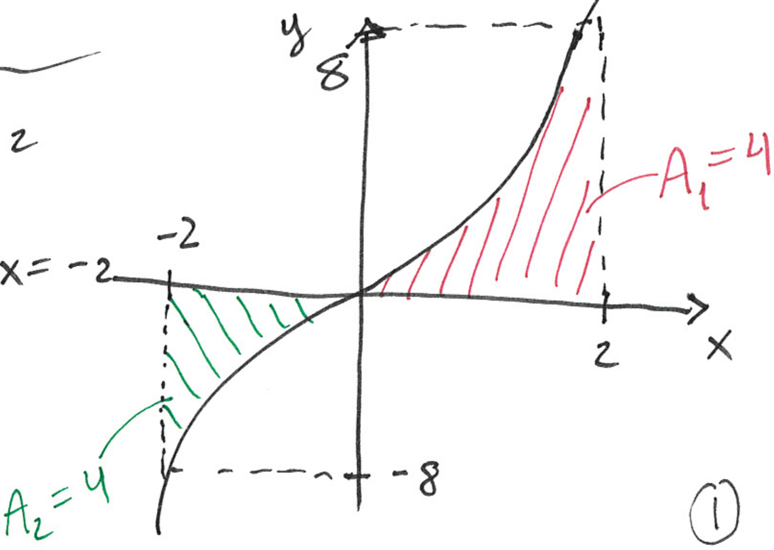
$a \to \infty \Rightarrow \frac{1}{a} \to 0$

APPLICATIONS OF INTEGRATION:

What about integration of functions that are not always ≥ 0?

Ex: $\displaystyle\int_{-2}^{2} x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=-2}^{2}$

$$= \frac{1}{4}\left(2^4 - (-2)^4\right)$$



$= \frac{1}{4}(16 - 16) = 0$ $\boxed{\text{Why?}}$ ⇝ Split:

$I_1 = \int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{x=0}^{2} = \frac{1}{4}(16-0)$

$= \underline{4}$

$I_2 = \int_{-2}^{0} x^3\,dx = \frac{1}{4}\left[x^4\right]_{x=-2}^{0} = \frac{1}{4}\left(0 - \overbrace{(-2)^4}^{16}\right)$

$= \frac{-16}{4} = \underline{-4}$

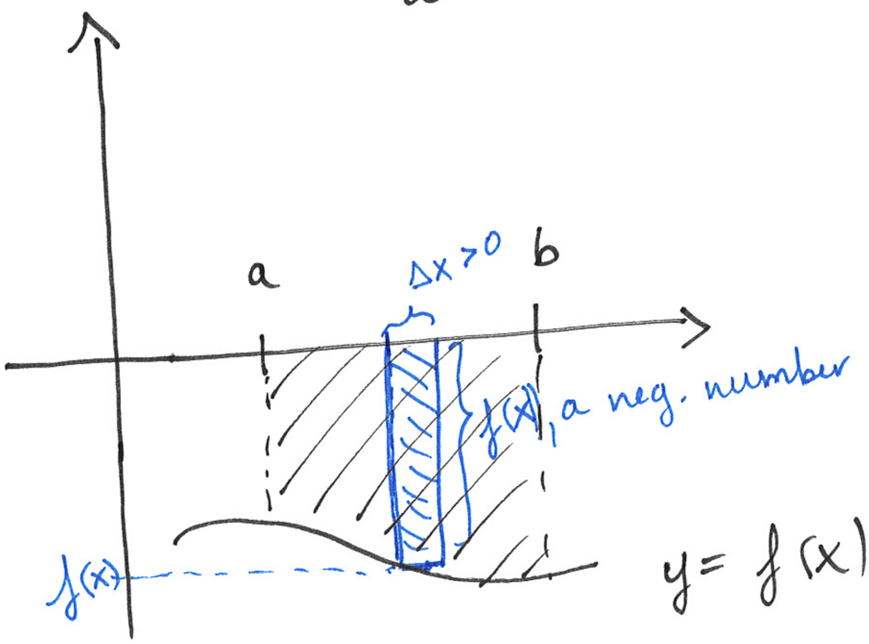
So: $I_1 + I_2 = 4 + (-4) = 0$

$A_1 - A_2$

When $f(x) \le 0$ in $[a, b]$:

Area between x-axis and graph of $y = f(x)$ in $[a, b]$ is:

$$A = \int_a^b - f(x)\,dx\,, \quad \text{so} \quad \int_a^b f(x)\,dx = -A$$



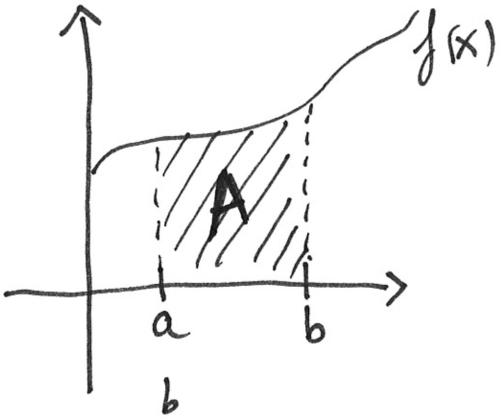
Height of rectangle:

$0 - f(x) = -f(x)$

a pos. number

3 cases:

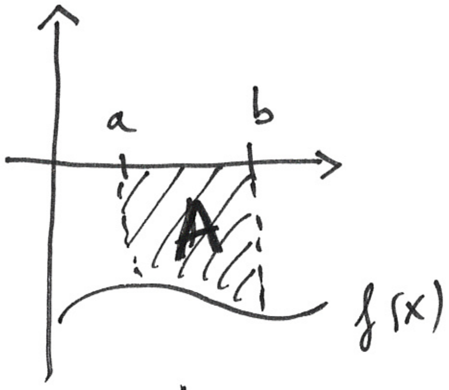
i) $f(x) \geq 0$



$$A = \int_a^b f(x)\,dx$$

(as before)

ii) $f(x) \leq 0$

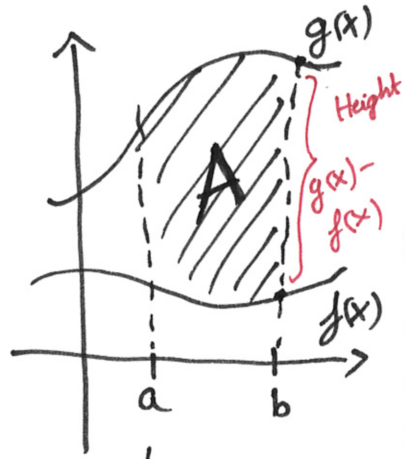


$$A = \int_a^b -f(x)\,dx$$

$$-A = \int_a^b f(x)\,dx$$

NB: An area is always non-negative

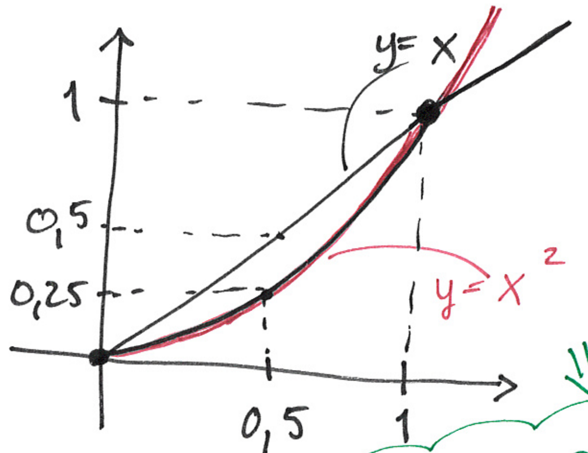
iii) $f(x) \leq g(x)$



$$A = \int_a^b (g(x) - f(x))\,dx$$

Ex: What is the area between $y = x$ and $y = x^2$ in $[0,1]$?

OBS: MAKE A FIGURE!!



OBS: On $[0,1]$, $x \geq x^2$

From iii)

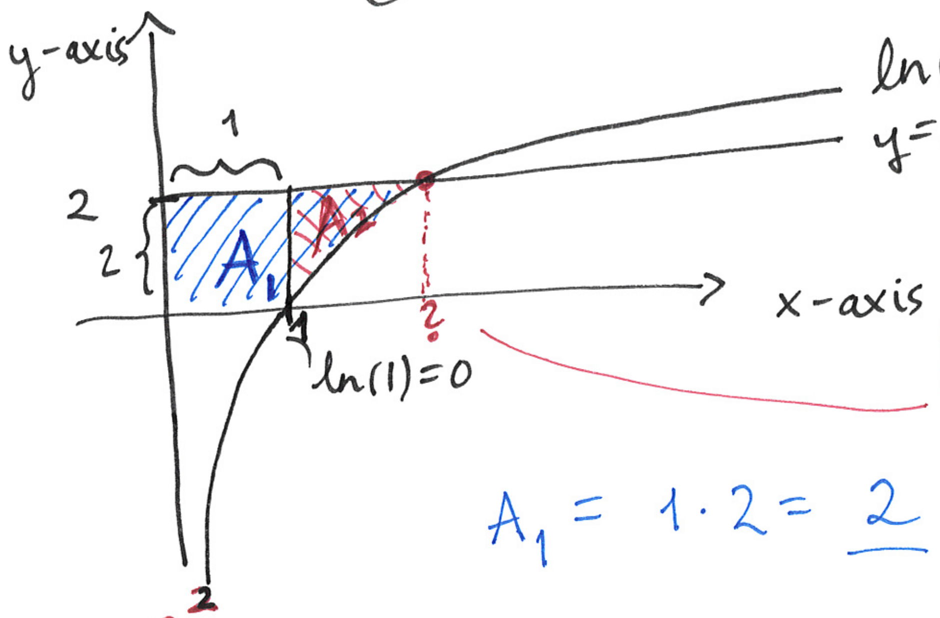
$$A = \int_0^1 x - x^2\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{x=0}^{1}$$

$$= \frac{1}{2}\cdot 1^2 - \frac{1}{3}\cdot 1^3 - \left(\frac{1}{2}\cdot 0^2 - \frac{1}{3}\cdot 0^3\right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \underline{\underline{\frac{1}{6}}} \quad (\approx 0{,}167)$$

Ex: Area bounded by $y = \ln x$, $y = 2$, the y-axis and the x-axis?



MAKE A FIGURE

What is this point?

$\ln x = 2$

$e^{\ln x} = e^2$

$x = e^2$

$A_1 = 1 \cdot 2 = \underline{2}$

$$A_2 = \int_1^{e^2} 2 - \ln x \, dx$$

$$\text{Area} = A_1 + A_2 = 2 + \int_1^{e^2} 2 - \ln x \, dx$$

int. by parts: $\ln x = 1 \cdot \ln x$

$$= 2 + \Big[2x - \underbrace{(x \ln x - x)} \Big]_{x=1}^{e^2}$$

$$= 2 + \Big[3x - x \ln x \Big]_{x=1}^{e^2}$$

$$= 2 + (3e^2 - e^2 \underbrace{\ln(e^2)}_{2}) - (3 \cdot 1 - 1 \cdot \underbrace{\ln 1}_{0})$$

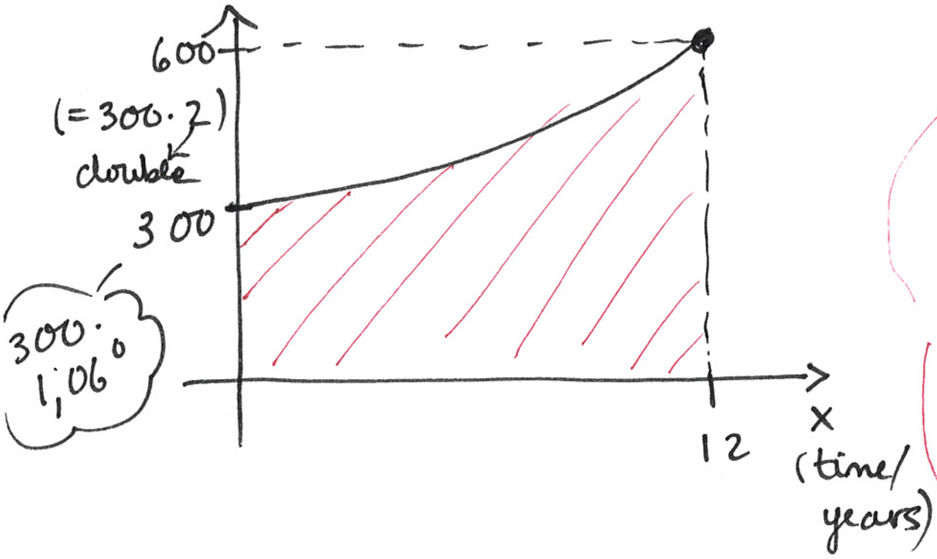
$$= 2 + 3e^2 - 2e^2 - 3$$

$$= \underline{\underline{e^2 - 1}} \quad (\approx 6{,}389)$$

Economic applications of the definite integral

Continuous cash flows:

<u>Ex</u>: $f(x) = 300 \cdot 1{,}06^x$ (cash flow in MNOK/year)



"Rule of 72"

Doubling takes approx.

$\frac{72}{6} = 12$ years

when money grows with 6% per year

<u>Total cash flow in 12 years</u> =

the area under the graph in [0, 12]

$$= \int_0^{12} f(x)\,dx = \int_0^{12} 300 \cdot 1{,}06^x\,dx$$

$\int a^x dx$

Integr. rule from 1st lecture

$$= \left[300 \cdot \frac{1{,}06^x}{\ln(1{,}06)}\right]_{x=0}^{12}$$

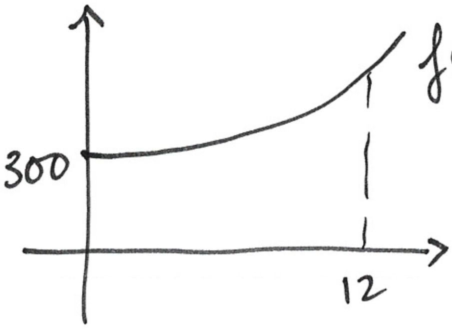
$$= \frac{300}{\ln(1{,}06)} \left[1{,}06^x\right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1,06)} \left(1,06^{12} - \underbrace{1,06^{0}}_{1} \right)$$

$$= \frac{300}{\ln(1,06)} \left(1,06^{12} - 1 \right) \approx \underline{5,211} \quad \text{MNOK}$$

Net present value of a continuous cash flow

NPV



$f(x) = 300 \cdot 1,06^{x}$, cash flow

ASSUME: r = discount rate $= 10\%$, continuous discounting

Money in future is worth less than money today: Inflation

NPV: $$\int_0^{12} \underbrace{f(x)}_{\text{cash flow}} \, e^{-rx} \, dx$$

$$= \int_0^{12} \overbrace{300 \cdot 1,06^{x}}^{f(x)} \cdot \overbrace{e^{-0,10x}}^{\text{discounting}} \, dx$$

$$= 300 \int_0^{12} \underbrace{1,06^{x}}_{''} \cdot e^{-0,10x} \, dx$$

$$= 300 \int_0^{12} e^{\ln(1,06)x} \, e^{-0,10x} \, dx$$

$$e^{\ln(1,06)x} = \left(e^{\ln(1,06)}\right)^{x} = 1,06^{x}$$

$$= 300 \int_0^{12} e^{(\ln(1,06) - 0,10)x} \, dx$$

$$= 300\left[\frac{1}{\ln(1{,}06)-0{,}10}\, e^{(\ln(1{,}06)-0{,}10)x}\right]_{x=0}^{12}$$

Substitution:

$$u = (\ln(1{,}06)-0{,}10)x$$

$$du = (\ln(1{,}06)-0{,}10)\,dx$$

$$\int e^{(\ln(1{,}06)-0{,}10)x}dx = \int e^u \frac{1}{\ln 1{,}06-0{,}10}du$$

$$= \frac{1}{\ln(1{,}06)-0{,}10} e^u + C$$

$$= \frac{300}{\ln(1{,}06)-0{,}10}\left(e^{(\ln(1{,}06)-0{,}10)\cdot 12} - \underbrace{e^0}_{1}\right)$$

$$\approx \underline{\underline{2{,}832}} \text{ MNOK}$$

FORMULAS: (Economic applications of the definite integrals)

Total cash flow: $\int_0^T f(x)\,dx$

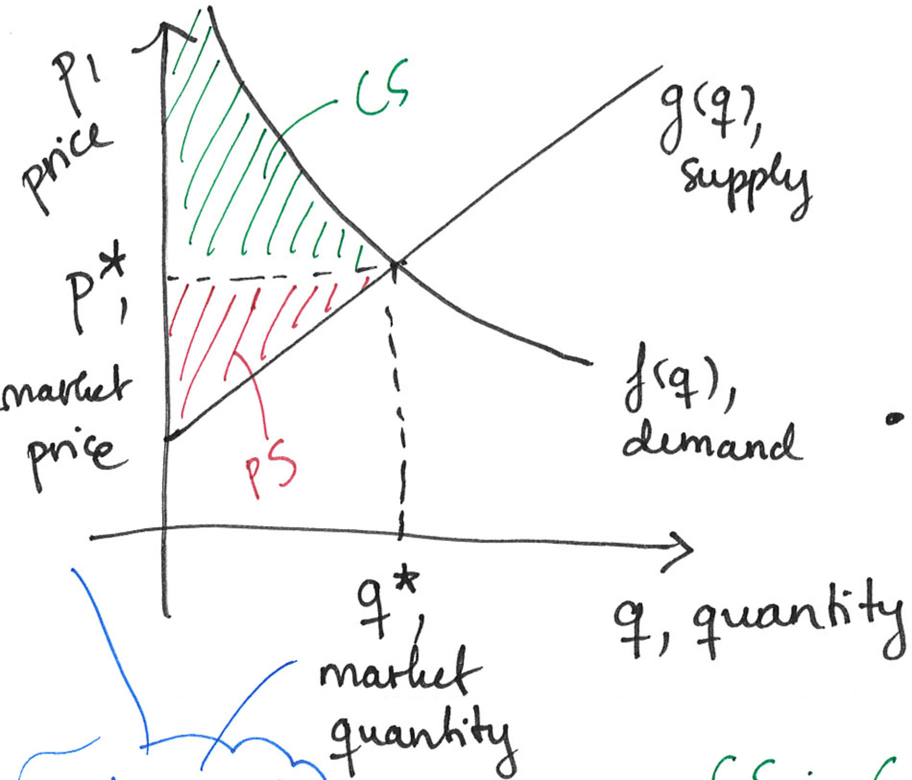
$f(x)$: cash flow per time unit

NPV of cash flow: $\int_0^T f(x)e^{-rx}\,dx$

r: discount rate

Consumer / producer surplus

$\underbrace{\text{Consumer}}_{CS}$ / $\underbrace{\text{producer surplus}}_{PS}$



$f(q^*) = g(q^*)$

- $p = f(q)$, demand function (inverse)
- $p = g(q)$, supply function (inverse)

CS : Consumer surplus

$$CS = \int_0^{q^*} f(q) - p^* \, dq$$

PS : Producer surplus

$$PS = \int_0^{q^*} p^* - g(q) \, dq$$

Ex: $f(q) = \dfrac{100}{q+5}$, $g(q) = q+5$

What is the market price/quantity?

$f(q) = g(q)$ "demand = supply"

$$\frac{100}{q+5} = q+5$$

$$100 = (q+5)^2$$

$$q+5 = \pm\sqrt{100} = \pm 10 = 10$$

quantity can't be negative

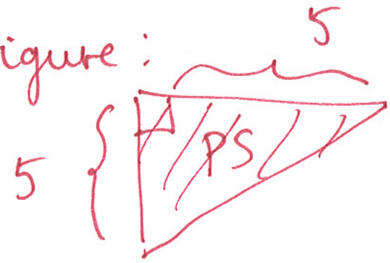
$$q = 10-5 = 5$$

$$\underline{q^* = 5}$$

$$p^* = q^* + 5 = 5+5 = \underline{10}$$

$$PS = \int_0^5 10 - (q+5)\,dq = \ldots = 12,5$$

OR: Area of triangle in figure:



CS:

$$CS = \int_0^5 \frac{100}{q-5} - 10\,dq$$

$$= \ldots = 100\ln 2 - 50$$