EBA 1180
Spring 24
Lecture 29
(5)

Recap question: $\int \frac{5}{4-9x^2}\,dx$

Plan?

$$5\int \frac{1}{(2-3x)(2+3x)}\,dx$$

Partial fractions: $\frac{1}{4-9x} = \frac{A}{2-3x} + \frac{B}{2+3x}$

$A = \ldots$, $B = \ldots$

Then substitution / ln-antidiff.

oo

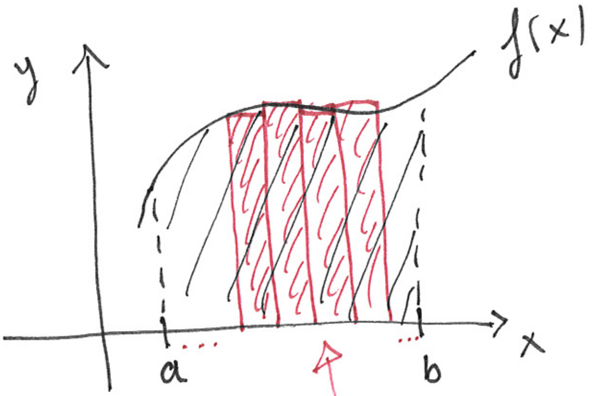
Definite integrals

Assume: i) f is a continuous function on $[a, b]$.

ii) $f(x) \geq 0$ for all x in $[a, b]$.

(iii) $a \leq b$)

Then,



$\int_a^b f(x)\,dx =$ the area under the graph of f in $[a, b]$

Can approximate with Riemann-sums

"Def" (definite integral):

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x).$$

So F is anti-derivative of f

Why?

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

↓ approx ↓

$$\sum g'(x)\,\Delta x = g(b) - g(a)$$

$$\sum \left(g(x+\Delta x) - g(x)\right) = g(b) - g(a)$$

$$g'(x) \approx \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$g(x+\Delta x) - g(x) \approx g'(x)\,\Delta x$$

sum of lots of small changes = total change

Ex:

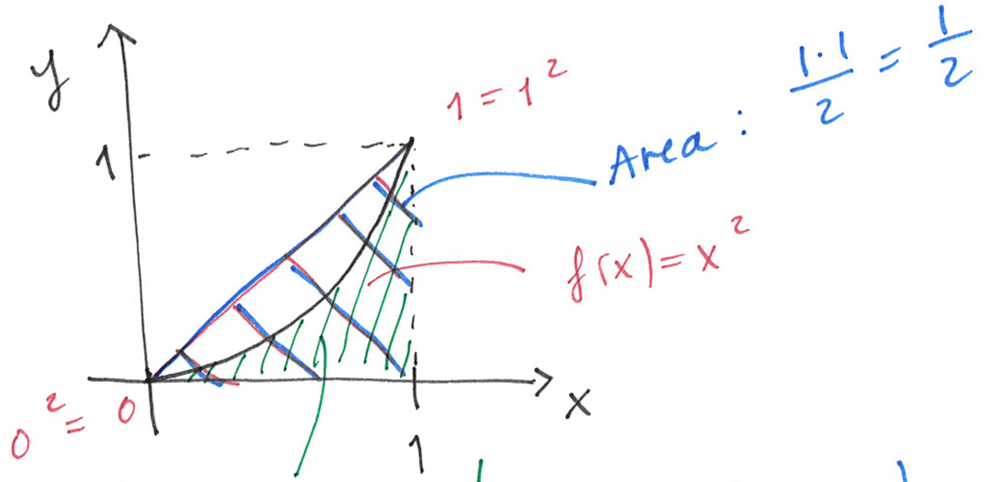
$$\int_0^1 \underbrace{x^2}_{f(x)}\,dx = \left[\overbrace{\tfrac{1}{3}x^3 + C}^{F(x)}\right]_{x=0}^{1}$$

$$= \underbrace{\left(\tfrac{1}{3}1^3 + C\right)}_{F(1)} - \underbrace{\left(\tfrac{1}{3}0^3 + C\right)}_{F(0)}$$

$$= \frac{1}{3} + \cancel{C} - \cancel{C} = \underline{\underline{\frac{1}{3}}}$$

Cancellation of constant always happens: won't write from now

Figure:



Area: $\int_0^1 x^2 \, dx = \frac{1}{3} < \frac{1}{2}$

Ex:

$$\int_1^2 \underbrace{\ln(x)}_{f(x)} \, dx = \Big[\underbrace{x \ln(x) - x}_{F(x)} \Big]_{x=1}^{2}$$

TRICK:

$\ln(x) = 1 \cdot \ln(x)$

$$\int \underbrace{\ln(x)}_{v} \cdot \underbrace{1}_{u'} \, dx = \underbrace{x}_{u} \underbrace{\ln(x)}_{v} - \int \underbrace{\cancel{x}}_{u} \underbrace{\frac{1}{\cancel{x}}}_{v'} \, dx$$

$u' = 1 \Rightarrow u = x$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$= x \ln(x) - x + C$

$$= \underbrace{(2 \ln(2) - 2)}_{F(2)} - \underbrace{(1 \cdot \ln(1) - 1)}_{F(1)}$$

$\ln 1 = 0$

$$= 2 \ln(2) - 2 - 0 + 1$$

$$= 2 \ln(2) - 1 \quad (\approx 0{,}386)$$

$\ln 1 = 0$

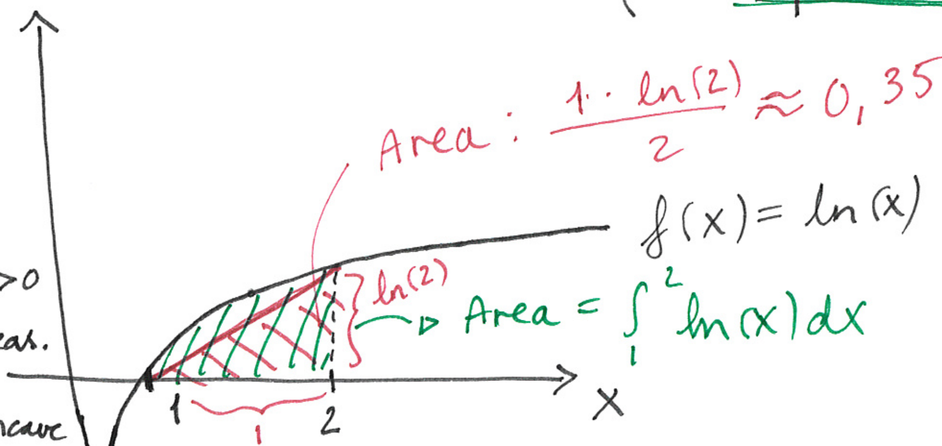
$\lim_{x \to +\infty} \ln x = \infty$

$\lim_{x \to 0^+} \ln x = -\infty$

$(\ln x)' = \frac{1}{x} > 0, \; x > 0$

$\Rightarrow \ln(x)$ is increas.

$(\ln x)'' = -\frac{1}{x^2} < 0$, concave



Ex:

$$\int_{x=0}^{x=1} x\sqrt{x^2+1}\,dx = \int_{u=1}^{u=2} \cancel{x}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du$$

$$u = x^2+1$$
$$du = 2x\,dx$$
$$dx = \frac{1}{2x}\,du$$
$$x=0 \Rightarrow u = 0^2+1 = 1$$
$$x=1 \Rightarrow u = 1^2+1 = 2$$

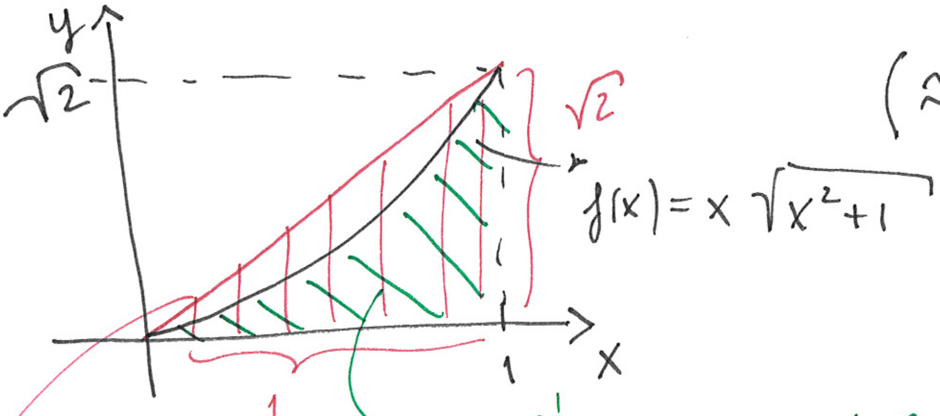
MIND THE INTEGRATION BOUNDS WHEN DOING SUBSTITUTION

$$= \int_1^2 \frac{1}{2} u^{\frac{1}{2}}\,du = \left[\frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=1}^{2}$$

$$= \left[\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\,\underbrace{u^{\frac{3}{2}}}_{=u\sqrt{u}}\right]_{u=1}^{2}$$

$$= \frac{2\sqrt{2}}{3} - \frac{1\sqrt{1}}{3} = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

$$(\approx 0{,}609)$$



Area $= \int_0^1 f(x)\,dx \approx 0{,}609$

Area: $\frac{1 \cdot \sqrt{2}}{2} \approx 0{,}7$

Alternative: Instead of inserting into:

$$\left[\frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=1}^{2} = \left[\frac{1}{2}\,\frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{x=0}^{1}$$

sub. back in for x

NB!

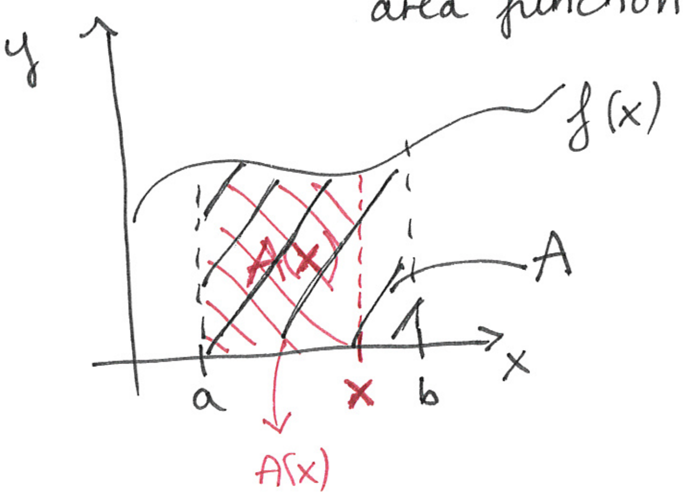
Theorem: If f is a continuous function on $[a,b]$ such that $f(x) \geqslant 0$ for x in $[a,b]$, then

the area under the graph of $f(x)$ in the interval $[a,b]$ $= \int_a^b f(x)\,dx = F(b) - F(a)$

where $F'(x) = f(x)$, so F is an anti-derivative of f.

Why? "Proof": Define

$\underbrace{A(x)}_{\text{area function}}$ = the area under $y = f(x)$ in $[a, x]$

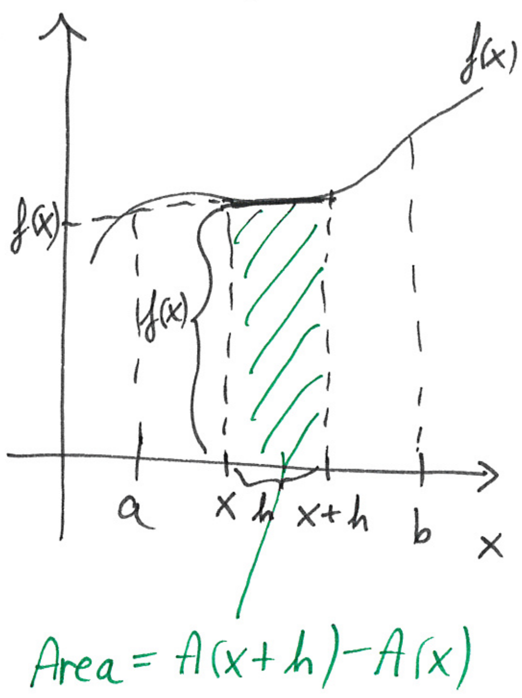


Also, let

A = area under $y = f(x)$ in $[a, b]$

Facts: $A(a) = 0$

$A(b) \overset{?}{=} A$



$$\underbrace{A'(x)} \approx \frac{A(x+h) - A(x)}{h}$$

def. of derivative

$$= \frac{\text{area of strip}}{h}$$

$$\approx \frac{f(x) \cdot \cancel{h}}{\cancel{h}} = \underbrace{f(x)}$$

So: $A'(x) \approx f(x)$, hence $A(x)$ is an anti-derivative of $f(x)$. But then,

$$\int_a^b f(x)\,dx = \left[A(x) \right]_{x=a}^{b} = A(b) - A(a)$$

$$= A - 0 = A$$

" ▩ "

Improper integrals

What if:

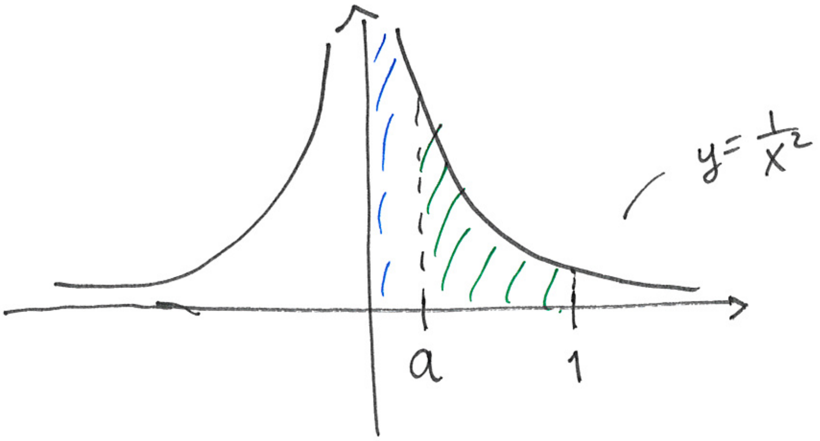
1) $f(x)$ is not continuous on $[a,b]$?

OR

2) $a = -\infty$ or $b = \infty$?

Ex: $$\int_0^1 \underbrace{\frac{1}{x^2}} dx \overset{\text{THIS MEANS:}}{=} \lim_{a \to 0} \int_a^1 \frac{1}{x^2}\, dx$$

$\frac{1}{x^2}$ is not defined for $x = 0$



<u>Why?</u>

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x\to \pm\infty} \frac{1}{x^2} = 0$$

$$\int_a^1 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{x=a}^{1} = \left[-\frac{1}{x}\right]_{x=a}^{1}$$

$$= -\frac{1}{1} - \left(-\frac{1}{a}\right) = -1 + \frac{1}{a}$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0} \int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0}\Big(-1+\underbrace{\frac{1}{a}}_{\to\infty}\Big)$$

$$= \underline{\underline{\infty}}$$