Recap

Integration methods

- Substitution:

$$u = g(x)$$
$$du = g'(x)\,dx$$
$$dx = \frac{1}{g'(x)}\,du$$

EBA 1180
Spring 24
Lect. 3 (27)

Thursday:
Problem set 27 (old exams) → Try yourself before class

See it's learning

- Integration by parts:

$$\int u'v\,dx = uv - \int uv'\,dx$$

Q: 1) $\int 5x^2 e^x\,dx$

2) $\int 5x^2 e^{x^3}\,dx$

Plan?

Ex:

$$\int u'\,v\,dx = u\cdot v - \int u\cdot v'\,dx$$

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{1}{\cancel{x}}\,dx$$

Int. by parts:

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$u' = x \Rightarrow u = \frac{1}{2}x^2$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\frac{1}{2}x^2 + C$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

$$\int v \cdot u' \, dx = u \cdot v - \int u \cdot v' \, dx$$

Ex:

$$\int 2x e^x \, dx = e^x 2x - \int e^x \cdot 2 \, dx$$

Int. by parts:

$u' = e^x \Rightarrow u = e^x$

$v = 2x \Rightarrow v' = 2$

$$= 2xe^x - 2\int e^x \, dx$$

$$= \underline{\underline{2xe^x - 2e^x + C}}$$

OBS: To check answer: Differentiate & see that you get integrand back

Does it work to choose opposite roles?

$$\int u' v \, dx = u \cdot v - \int u \cdot v' \, dx$$

$$\int 2x e^x \, dx = x^2 e^x - \underbrace{\int x^2 e^x \, dx}$$

Int. by parts:

$u' = 2x \Rightarrow u = x^2$

$v = e^x \Rightarrow v' = e^x$

Looks worse! Doesn't work with opposite roles.

Ex:

$$\int \ln x \, dx = \int \underbrace{1 \cdot \ln x}_{} \, dx \qquad \int u' v \, dx$$

TRICK!

Int. by parts:

$u' = 1 \Rightarrow u = x$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$$uv - \int u v' \, dx$$

$$= x \ln x - \int \cancel{x} \frac{1}{\cancel{x}} \, dx$$

$$= \underline{\underline{x \ln x - x + C}}$$

FORMULA: $\int \ln x \, dx = x \ln x - x + C$

$\int v\, u' \, dx = v \cdot u - \int v' u \, dx$

Ex: $\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$ Int. by parts: Did earlier today!

Int. by parts:

$u' = e^x \Rightarrow u = e^x$

$v = x^2 \Rightarrow v' = 2x$

$= x^2 e^x - (2x e^x - 2e^x) + C$

$= x^2 e^x - \underline{\underline{2x e^x + 2e^x}} + C$

METHOD: Integration of rational functions / fractions

Ex: i) $\int \frac{2}{1-x} \, dx$ ii) $\int \frac{2x}{1-x^2} \, dx$

iii) $\int \frac{2}{1-x^2} \, dx$

Q: Discuss & guess: Which technique? Plan?

What to do?

i): $\int \frac{2}{1-x} \, dx = \int \frac{2}{u} (-du)$

$u = 1-x$
$du = -dx$
$dx = -du$

$= -2 \int \frac{1}{u} \, du$

$$= -2 \ln|u| + C$$

$$= -2 \ln|1-x| + C$$

In general: If $a \neq 0$:

$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\frac{1}{a}\,du$$

$u = ax+b$
$du = a\,dx$
$dx = \frac{1}{a}\,du$

$$= \frac{A}{a}\int \frac{1}{u}\,du$$

$$= \frac{A}{a}\ln|u| + C$$

$$= \frac{A}{a}\ln|ax+b| + C$$

FORMULA:

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C, \quad a \neq 0$$

Ex (alt.): $\int \frac{x}{1-x}\,dx = \int -1 + \frac{1}{1-x}\,dx$

$$= -x - \ln|1-x| + C$$

Polynomial division:

$x : (-x+1) = -1 + \frac{1}{1-x}$
$-(x-1)$
1 ← Remainder

Polynomial div: If degree numerator $\geq$ degree denominator

Type ii):

SUBSTITUTION

$$\int \frac{2x}{1-x^2}\,dx = \int \frac{\cancel{2x}}{u}\left(-\frac{1}{\cancel{2x}}\right)du$$

$u = 1-x^2$

$du = -2x\,dx$

$dx = -\frac{1}{2x}\,du$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|1-x^2| + C$$

Type iii):

$$\int \frac{2}{1-x^2}\,dx = \int \frac{\cancel{2}}{u}\,\frac{1}{-\cancel{2}x}\,du$$

Substitution?

$u = 1-x^2$

$du = -2x\,dx$

$dx = -\frac{1}{2x}\,du$

$$= -\int \underbrace{\frac{1}{xu}}_{\text{Looks bad!}}\,du$$

Actually: Not solvable by substitution. Instead:

METHOD: Partial fractions

Norwegian: Delbröksoppspaltning

Ex: $$\frac{2}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$

AIM: Find A and B s.t.

Factorize denominator:
$$1-x^2 = (1-x)(1+x)$$

• $(1-x^2)$

NB: A and B are unknown constants

$= (1+x)(1-x)$

$$2 = \frac{A}{\cancel{1+x}}\cancel{(1+x)}(1-x) + \frac{B}{\cancel{1-x}}(1+x)\cancel{(1-x)}$$

$$2 = A(1-x) + B(1+x)$$

$$2 = A - Ax + B + Bx$$

$$0 \cdot x + 2 = (B-A)x + (A+B)$$

Compare coefficients

1) $B - A = 0 \Rightarrow B = A$

2) $A + B = 2$

Insert

$2A = 2$

$\underline{A = 1} \Rightarrow \underline{B = 1}$

Hence, $$\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}$$