EBA 1180
Lecture 2 (26)
Spring 24

Integration

Ex: $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx$

"TRICK!"

$= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

power rule

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2}{3} x^{\frac{3}{2}} + C$

$\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$= \dfrac{2}{3} x\sqrt{x} + C$

$x^{\frac{3}{2}} = x^{1+\frac{1}{2}} = x\, x^{\frac{1}{2}} = x\sqrt{x}$

Ex: $\int \dfrac{1}{x^2}\, dx = \int x^{-2}\, dx = \dfrac{x^{-2+1}}{-2+1} + C$

$= \dfrac{x^{-1}}{-1} + C$

$= -\dfrac{1}{x} + C$

Ex: $\int \dfrac{x^2 - 2x + 3}{x}\, dx = \int \dfrac{x^2}{x} - \dfrac{2x}{x} + \dfrac{3}{x}\, dx$

$= \int x - 2 + 3 \cdot \dfrac{1}{x}\, dx$

$$= \frac{1}{2}x^2 - 2x + 3\ln|x| + C$$

power rule, ii) & iii)

Substitution

Exp. rule: $\int e^u\,du = e^u + C$

Ex: $\int e^{2x}\,dx = \int e^{u}\,dx$ → PROBLEM: Not same variable. Need du

make e^{2x} look like e^u

$2x = u$

$\boxed{u' = 2}$

$\frac{du}{dx} = u'$

$du = u'\,dx$

$dx = \frac{du}{u'}$

$$= \int e^u \frac{1}{\underbrace{u'}_{2}}\,du$$

$$= \int e^u \frac{1}{2}\,du = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{2x} + C$$

CHECK: $\left(\frac{1}{2}e^{2x} + C\right)' = \frac{1}{2}e^{2x}\cdot(2x)' + 0$

CHAIN RULE

$$= \frac{1}{\cancel{2}}e^{2x}\cdot\cancel{2} = e^{2x}$$

original integrand ⇒ OK!

FORMULA: (Substitution)

$$du = u'\,dx$$

$$dx = \frac{1}{u'}\,du$$

where u' means the derivative of u wrt. x

Ex: $\int x\sqrt{x^2+1}\,dx$

$= \int \cancel{x}\sqrt{\underbrace{u}_{x^2+1}}\,\frac{1}{2\cancel{x}}\,du$

Ugly core? Yes! x^2+1

$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ dx &= \frac{1}{2x}\,du \end{aligned}}$

TRICK

$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\,u^{\frac{3}{2}} + C$

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

Ex: $\int x e^{-x^2}\,dx = \int \cancel{x}e^{u}\left(-\frac{1}{2\cancel{x}}\right)du$

$u = -x^2$

$du = -2x\,dx$

$dx = -\frac{1}{2x}\,du$

$= -\frac{1}{2}\int e^{u}\,du$

$= -\frac{1}{2}e^{u} + C$

$= -\frac{1}{2}e^{-x^2} + C$

EXTRA: $\int \frac{\ln x}{x}\,dx$

Extra extra: $\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx$

Ex: $\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\,\cancel{x}\,du$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$= \int u\,du$

$= \frac{1}{2}u^2 + C$

$$= \frac{1}{2}(\ln x)^2 + C$$

Ex: $\displaystyle\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}(-2\cancel{\sqrt{x}})\,du$

$u = 1-\sqrt{x}$
$= 1 - x^{\frac{1}{2}}$

$du = -\frac{1}{2}x^{-\frac{1}{2}}dx$
$= -\frac{1}{2\sqrt{x}}dx$

$dx = -2\sqrt{x}\,du$

$= -2\int e^{u}\,du$

$= -2e^u + C$

$= -2e^{1-\sqrt{x}} + C$

Ex: $\displaystyle\int e^{\sqrt{x}}\,dx = \int e^u\, 2\sqrt{x}\,du$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\,du$

$= 2\int e^u \cdot u\,du$

Product $\rightsquigarrow$ Integration by parts

(corresponds to product rule for differentiation)

$$= 2\left(ue^u - e^u\right) + C$$

Integration by parts

$$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$= 2e^{\sqrt{x}}\left(\sqrt{x} - 1\right) + C$$

METHOD: Integration by parts

FORMULA: $$\int u' \cdot v \, dx = uv - \int u v' \, dx$$

Functions of x

Hence, in example: $\int \underbrace{x}_{v} \cdot \underbrace{e^x}_{u'} \, dx$

1) Decide which factor can most easily be anti-diff'ed. Call this u'.
   Ideally: v' is simple.

2) Anti-diff u': Get u
   Differentiate v: Get v'

   Ex: $u' = e^x \Rightarrow u = e^x$
   $v = x \Rightarrow v' = 1$

3) Use formula:

   Ex: $\int e^x x \, dx = e^x x - \int e^x \cdot 1 \, dx$

   $= xe^x - \underbrace{\int e^x \, dx}_{\text{EASY!}}$

$$= xe^x - e^x + C$$

Why does integration by parts hold?

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

PRODUCT RULE FOR DIFFERENTIATION

$\Downarrow$ Integrate on both sides

$$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' dx$$

$$\Downarrow$$

So,

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' dx}$$