EBA 1180, lecture 20, 1 Nov. 2023, Runar Ile

Plan: 1. Implicit differentiation
2. The second derivative and curvature
3. Convex optimization

1. Implicit differentiation

Ex A curve is implicitly defined by the equation $\boxed{y^2 - x^3 = 1}$

a) Express y' by x and y using implicit differentiation

b) Find all solutions for y when $x = 2$

c) Compute y' for these points

d) Find the function expression(s) for the tangent line(s) at $x = 2$

Solution

a) $(y^2)'_x - (x^3)'_x = (1)'_x$

Chain rule: $u = u(x) = y$ and $g(u) = u^2$

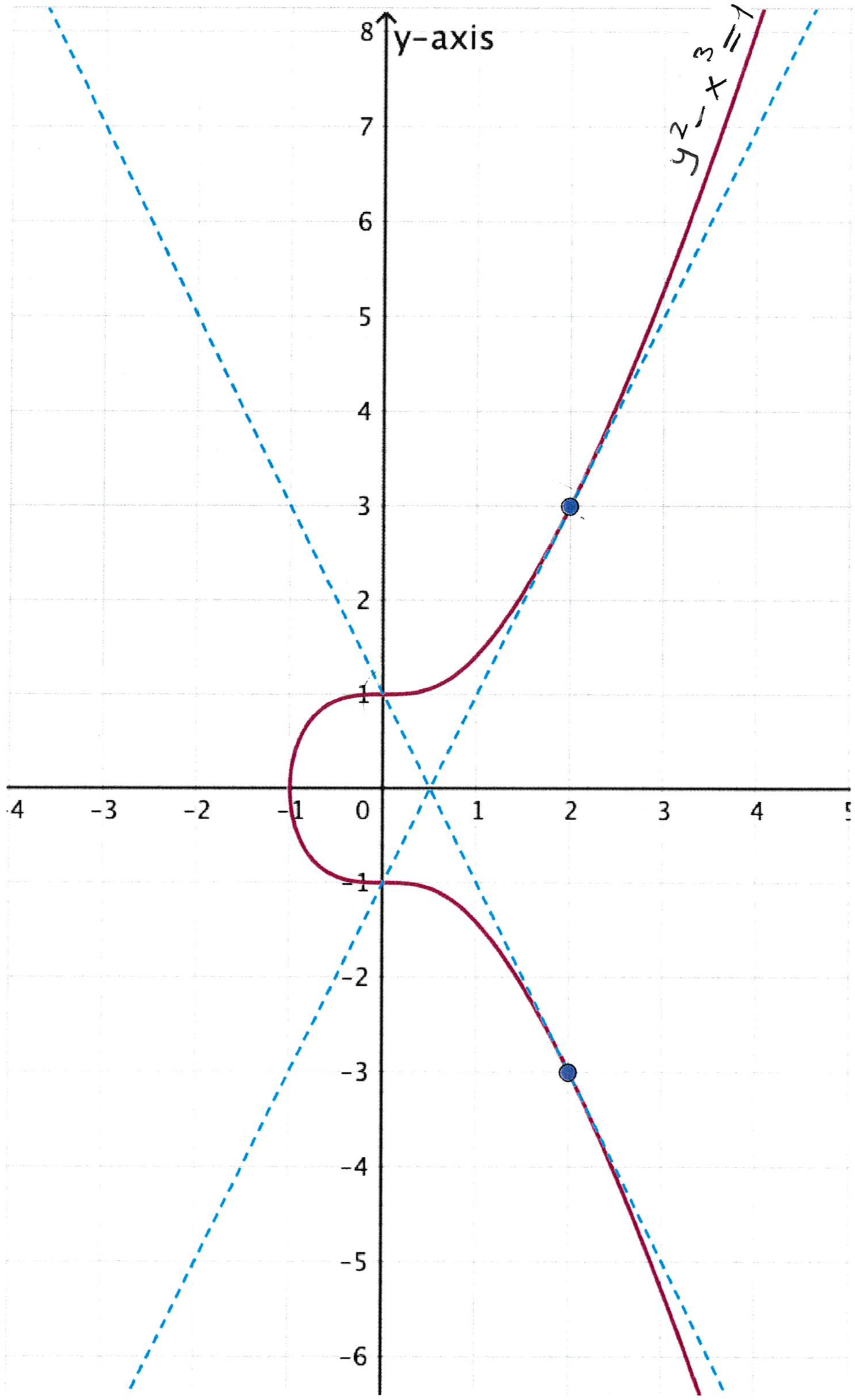
$u'(x) = y'_x$ $\qquad g'(u) = 2u$

So $(y^2)'_x = 2u \cdot u'_x = 2y \cdot y'$

$2y \cdot y' - 3x^2 = 0$ – solve for y'

$2y \cdot y' = 3x^2$ $\quad | : 2y$

$$\boxed{y' = \frac{3x^2}{2y}}$$


y-axis
$y^2 - x^3 = 1$
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-4
-3
-2
-1
0
1
2
3
4
5

b) $\underline{x=2}$, solve $y^2 - 2^3 = 1$

$$y^2 = 1 + 8 = 9$$

$$\underline{\underline{y = \pm 3}}$$

c) $(2,3)$: $y' = \frac{\cancel{3} \cdot 2^{\cancel{2}}}{\cancel{2} \cdot \cancel{3}} = \underline{\underline{2}}$

$(2,-3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot (-3)} = \underline{\underline{-2}}$

d) Tangent line through $(2,3)$:

$$h(x) - 3 = 2 \cdot (x - 2)$$

so $\underline{\underline{h(x) = 2x - 1}}$

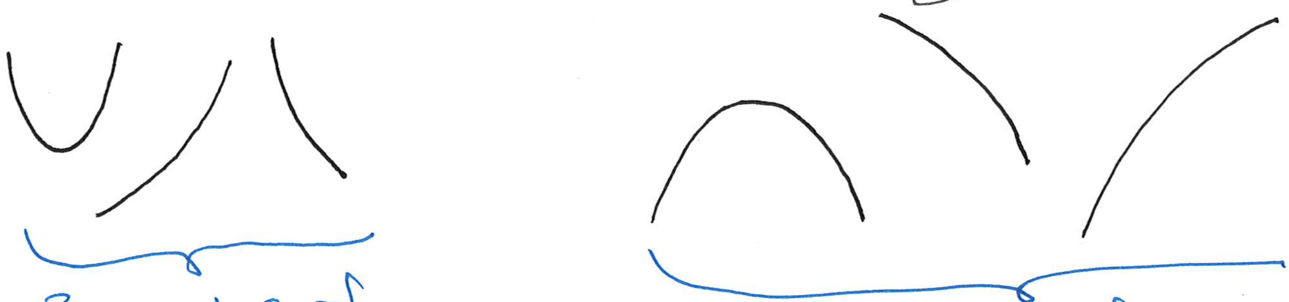
Tangent line through $(2,-3)$:

$$g(x) - (-3) = -2 \cdot (x - 2)$$

so $\underline{\underline{g(x) = -2x + 1}}$

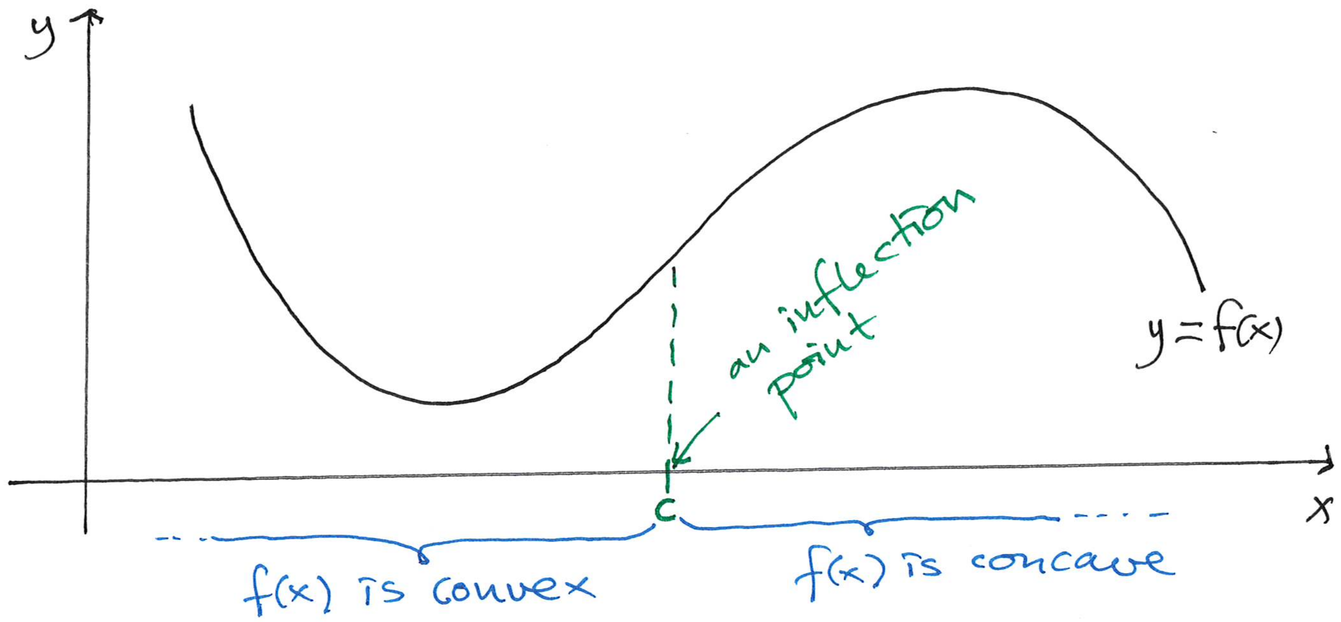
2. The second order derivative and curvature

bending up | bending down



3 graphs of convex functions

3 graphs of concave functions



Definition

- f(x) is convex in the interval $[a, b]$ if $f''(x) \geq 0$ for all x in $\langle a, b \rangle$.
- f(x) is concave in the interval $[a, b]$ if $f''(x) \leq 0$ for all x in $\langle a, b \rangle$.
- A number c is an inflection point for f(x) if $f''(x)$ changes sign at $x = c$.

Start : 11.03

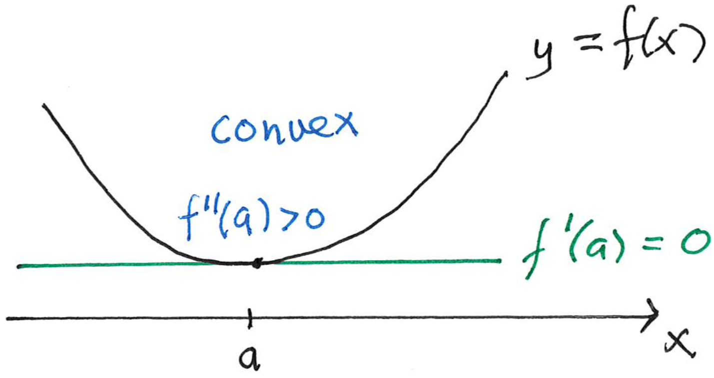
Note If $f(x)$ is convex, then $f'(x)$ is an increasing function.

If $f(x)$ is concave, then $f'(x)$ is a decreasing function.

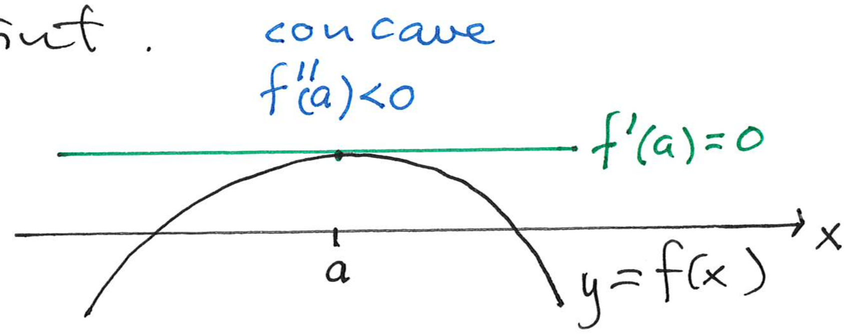
second derivative test (Sec. 8.5)

Suppose $x = a$ is a stationary point for $f(x)$.

If $f''(a) > 0$ then $x = a$ is a (local) minimum point.



If $f''(a) < 0$ then $x = a$ is a (local) maximum point.



Ex $f(x) = x^3 - 3x^2 + 5$ Find local max/min. points by using the second derivative test.

Solution calculate $f'(x) = 3x^2 - 6x$.

Stationary points: solutions of the eq. $f'(x) = 0$

e.g. $3x^2 - 6x = 0$ $\quad$ $3x$ is a common factor

$3x(x-2) = 0$

either $\underline{x = 0}$ or $\underline{x = 2}$

Calculate $f''(x) = [f'(x)]' = (3x^2 - 6x)'$

$$= \underline{6x - 6}$$

$f''(0) = 6 \cdot 0 - 6 = -6 < 0$

so $x = 0$ is a (local) maximum point.

$f''(2) = 6 \cdot 2 - 6 = 6 > 0$

so $x = 2$ is a (local) minimum point.

---

3. Convex optimization

Fact
- If $f(x)$ is convex everywhere in its domain (an interval), then any stationary point is a global minimum point.

- If $f(x)$ is concave everywhere in its domain (an interval), then any stationary point is a global maximum point.

Ex $f(x) = x^4 + 5x^2 + 3$, $D_f = \langle \leftarrow, \rightarrow \rangle = \langle \infty, \infty \rangle = \mathbb{R}$

a) Find the stationary points

b) Determine if they are global max. or min. points.

c) Determine the extremal values.

Solution

a) Calculate $f'(x) = 4x^3 + 10x$

Stationary points are solutions to the eq.

$$4x^3 + 10x = 0 \quad \text{(x is a common factor)}$$

$$x(4x^2+10) = 0$$

so $\underline{\underline{x=0}}$ is the only stationary point.

b) Calculate $f''(x) = (4x^3+10x)' = 12x^2 + 10$

which is greater or equal to 10

for all x. So $x=0$ is a global min. point.

(The graph of $f(x)$ is convex.)

c) $f(0) = 0^4 + 5 \cdot 0^2 + 3 = \underline{\underline{3}}$ is the global minimum value.