EBA1180, lecture 13, 10 Oct. 2023, Runar Ile

Plan 1. Rational functions and asymptotes

2. Hyperbolas

1. Rational functions and asymptotes

Rational function $f(x) = \frac{p(x)}{q(x)}$ polynomials

<u>Ex</u> $f(x) = \frac{2x+1}{x^2+3}$ - would like to see what happens when x is big.

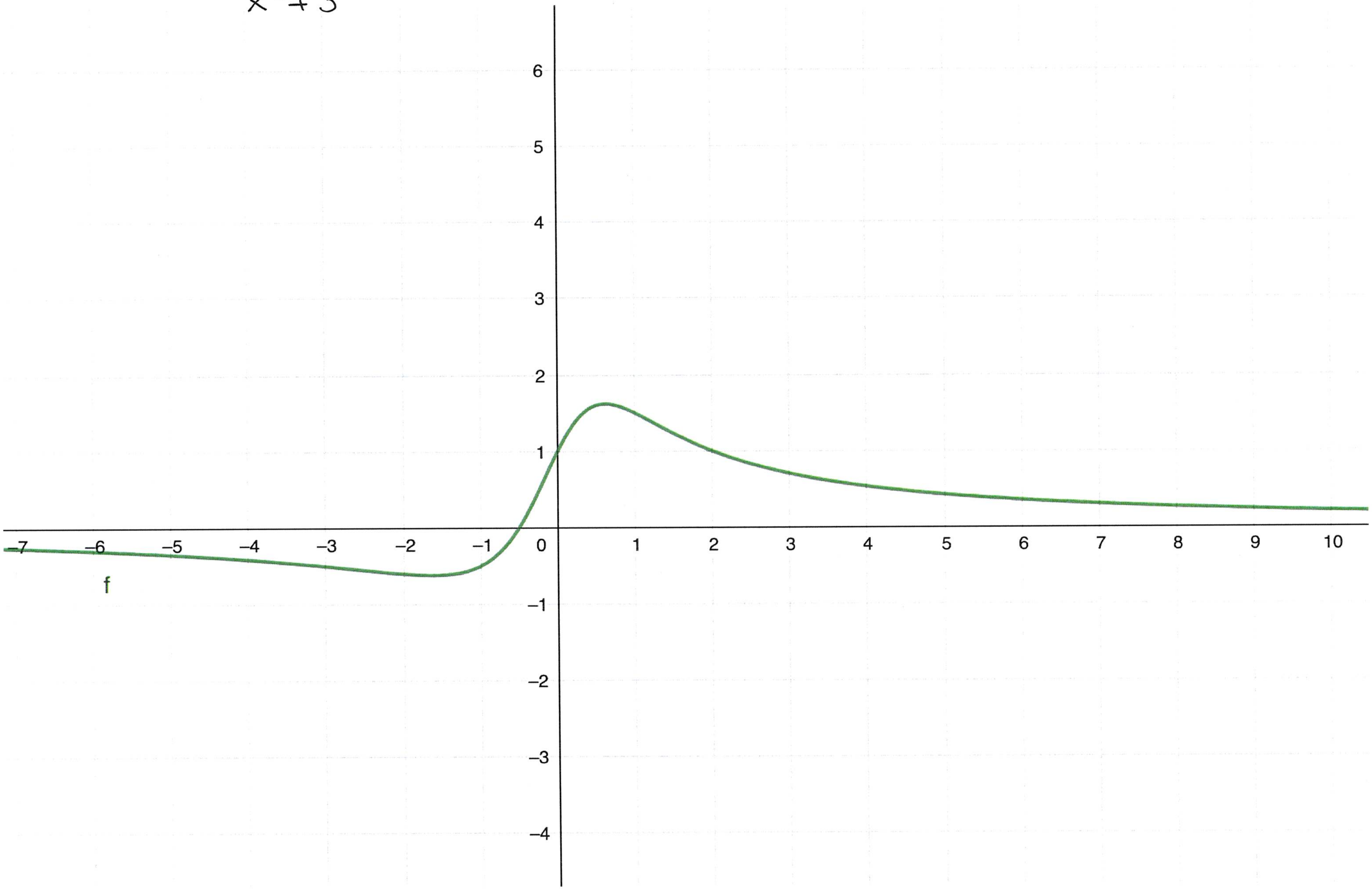
- divide by x^2 both in the numerator and in the denominator

$$\frac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \xrightarrow[x \to \pm\infty]{} \frac{0}{1} = 0$$

$$f(1000) = \frac{\frac{2}{1000} + \frac{1}{1000^2}}{1 + \frac{3}{1000^2}} = 0.00200099\ldots$$

This means that the line $y = 0$ (x free) is a horizontal asymptote for $f(x)$. The graph of $f(x)$ is approaching the x-axis (the horizontal asymptote) when x becomes big (pos/neg.).

Ex $f(x) = \frac{2x+1}{x^2+3}$



Ex $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$ $(x \neq 1, x \neq 5)$

What happens when x is approaching 1 or 5 ?

If $x \to 1^-$ "x is approaching 1 from below"

$x = 0.99$, $x = 0.999$, $x = 0.9999$...

then

$$\left.\begin{array}{l} x-1 \to 0^- \\ x-5 \to -4^- \\ 2x+1 \to 3^- \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{2x+1}^{\to 3^-}}{\underbrace{(x-1)}_{0^-}\underbrace{(x-5)}_{-4^-}} \xrightarrow[x \to 1^-]{} +\infty$$

If $x \to 1^+$ "x is approaching 1 from above"

$x = 1.1$, $x = 1.01$, $x = 1.001$

then

$$\left.\begin{array}{l} x-1 \to 0^+ \\ x-5 \to (-4)^+ \\ 2x+1 \to 3^+ \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{2x+1}^{\to 3^+}}{\underbrace{(x-1)}_{0^+}\underbrace{(x-5)}_{(-4)^+}} \xrightarrow[x \to 1^+]{} -\infty$$

Conclusion The line $x=1$ (y free) is a vertical asymptote for $f(x)$

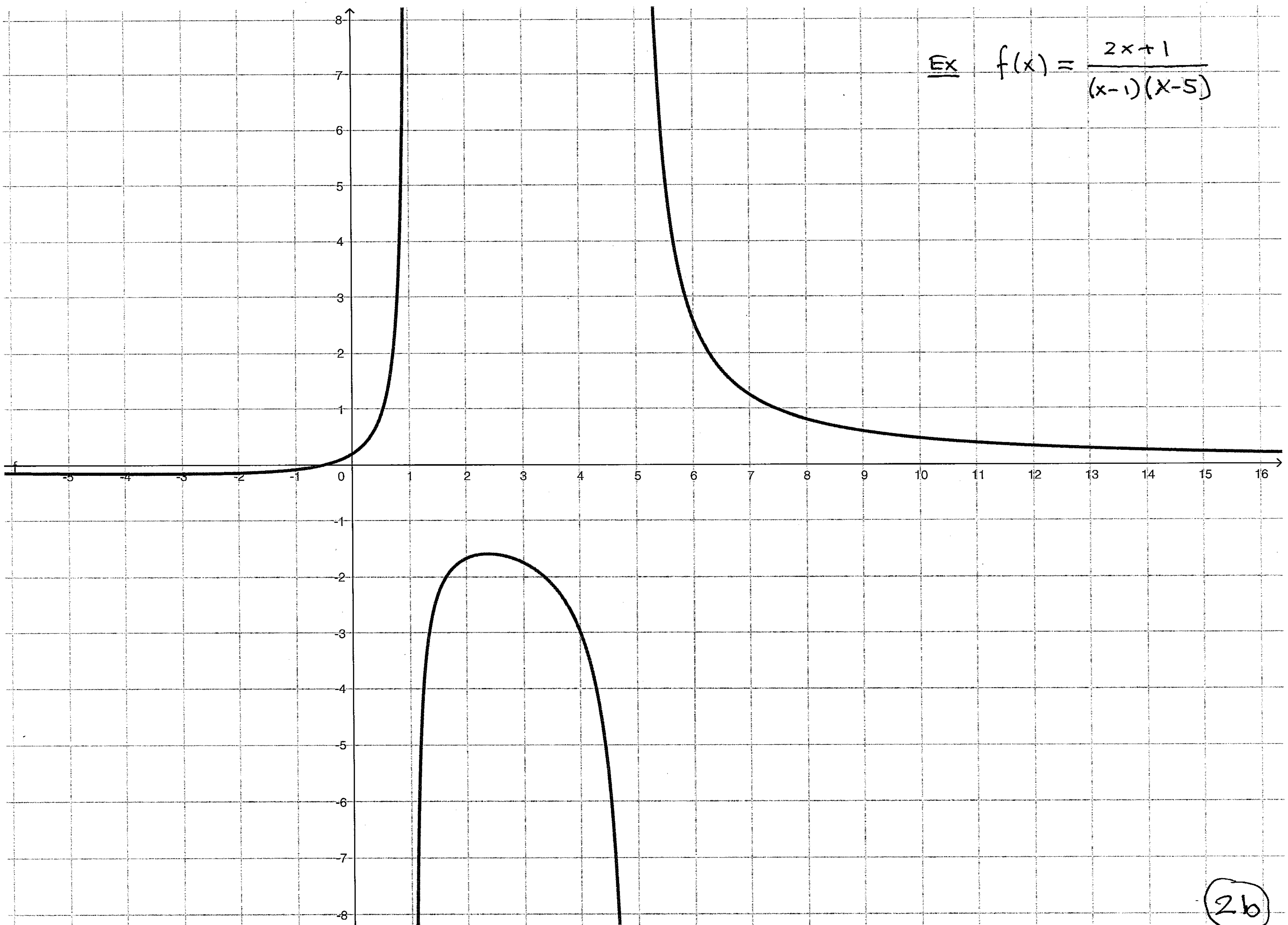
The graph of $f(x)$ is approaching the vertical line $x=1$ when $x \to 1$.

Note The line $x=5$ (y free) is another vertical asymptote for $f(x)$:

$$f(x) \xrightarrow[x \to 5^-]{} -\infty, \quad f(x) \xrightarrow[x \to 5^+]{} +\infty$$

$f(x)$ also has a horizontal asymptote $y=0$

__Ex__ $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$



2b

Non-vertical (oblique) asymptotes

Ex $f(x) = x-5+\dfrac{2}{x-4}$ has vertical asymptote $x=4$

But also an oblique asymptote:

Put $g(x) = x-5$.

Then the graph of $f(x)$ is approaching the graph of $g(x)$ when $x \longrightarrow \pm\infty$ because

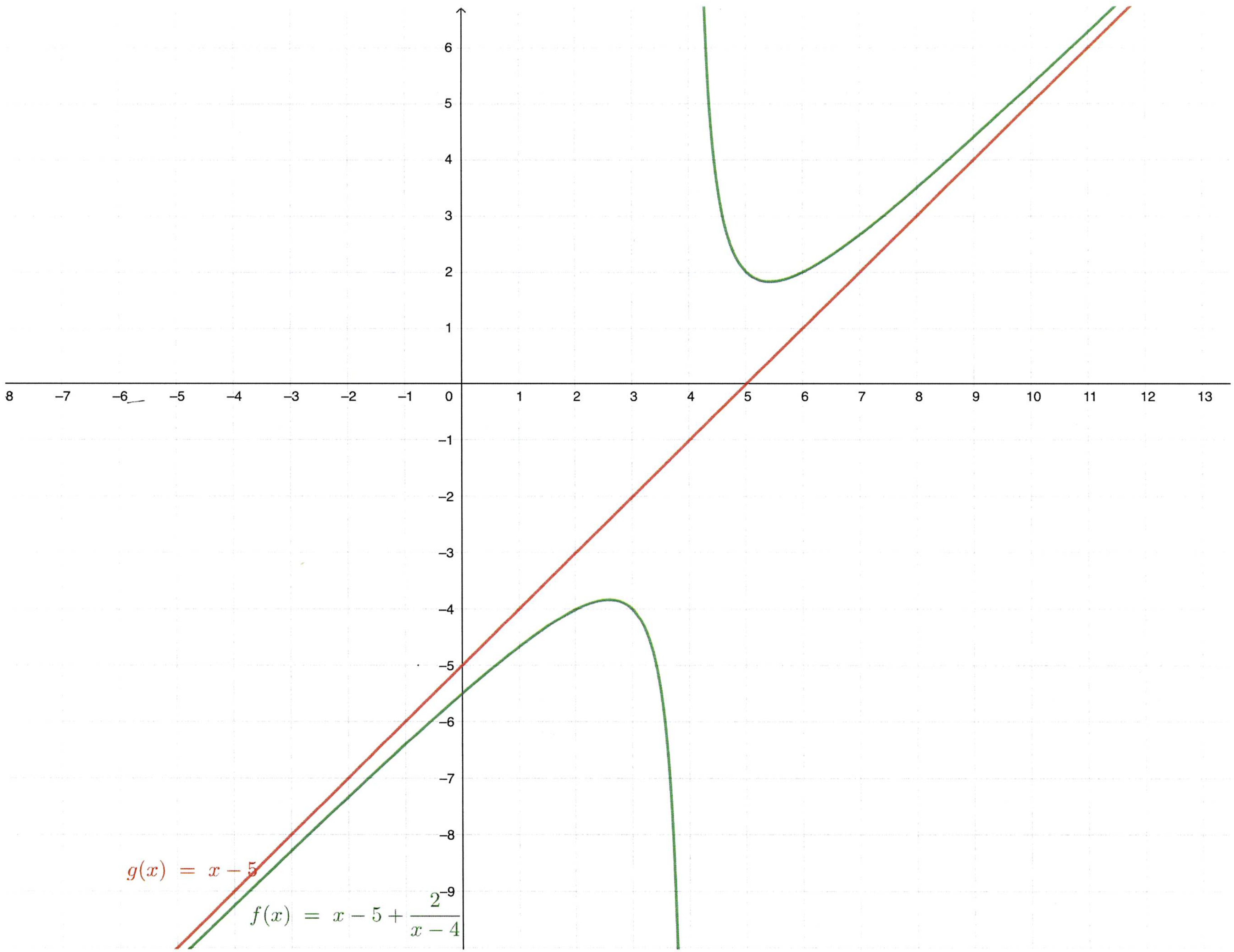
$$f(x)-g(x) = \frac{2}{x-4} \xrightarrow[x\to\pm\infty]{} 0$$

Note that $f(x) = \dfrac{(x-5)(x-4)+2}{x-4} = \dfrac{x^2-9x+22}{x-4}$

- have to do polynomial division to find the better expression $\underbrace{(x-5)}_{= g(x)} + \dfrac{2}{x-4}$

The graph of $g(x)$ is a non-vertical asymptote for $f(x)$.

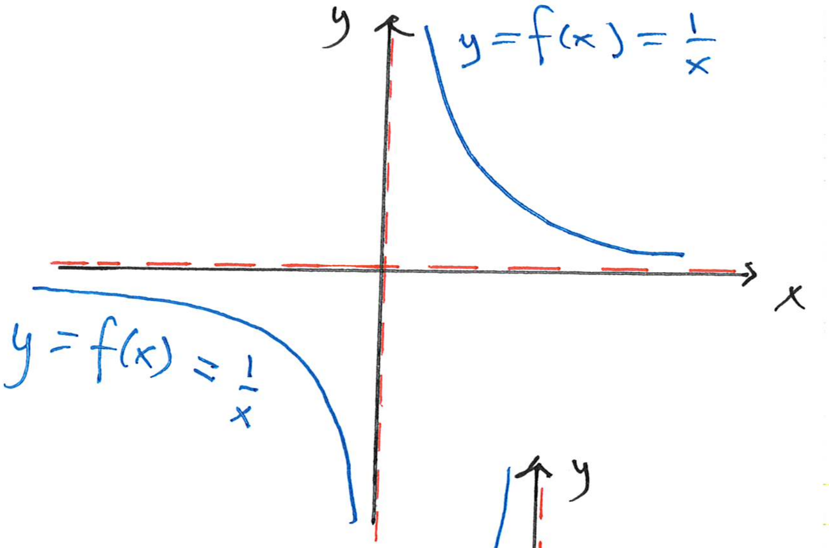
Start: 11.12



6
5
4
3
2
1
8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10 11 12 13
−1
−2
−3
−4
−5
−6
−7
−8
−9
g(x) = x − 5
f(x) = x − 5 + 2/(x − 4)

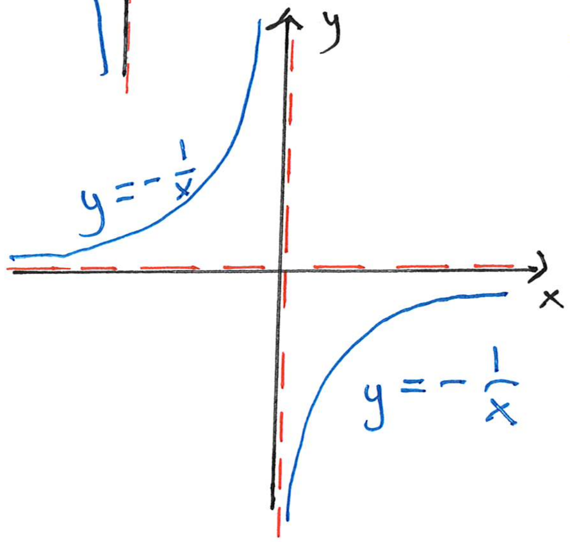
3b

2. Hyperbolas



EX $f(x)=\frac{1}{x}$ $(x\neq 0)$

The line $y=0$ is a horizontal asymptote.
The line $x=0$ is a vertical asymptote



EX $f(x)=-\frac{1}{x}$ $(x\neq 0)$

– the same asymptotes.

Definition A function $f(x)$ is a hyperbola function if it can be written as

$$f(x)=c+\frac{a}{x-b} \quad (a\neq 0)$$

EX $f(x)=\frac{3x-5}{x-2}$ is a hyperbola function because polynomial division gives

$$(3x-5):(x-2)=3+\frac{1}{x-2} \quad \text{so} \quad \begin{array}{l} a=1 \\ b=2 \\ c=3 \end{array}$$

$$\begin{array}{l} -(3x-6) \quad \leftarrow \cdot(x-2) \\ \hline \quad 1 \end{array}$$

So $f(x)=3+\frac{1}{x-2}$.

We have

$$3+\frac{1}{x-2}\xrightarrow[x\to 2^-]{}-\infty \quad \text{and} \quad 3+\frac{1}{x-2}\xrightarrow[x\to 2^+]{}+\infty$$

So the line $x = 2$ is a vertical asymptote.

Also note that $3 + \frac{1}{x-2} \xrightarrow[x \to \pm\infty]{} 3^{\pm}$

so the line $y = 3$ is a horizontal asymptote

$f(1) = 3 + \frac{1}{1-2} = 2$

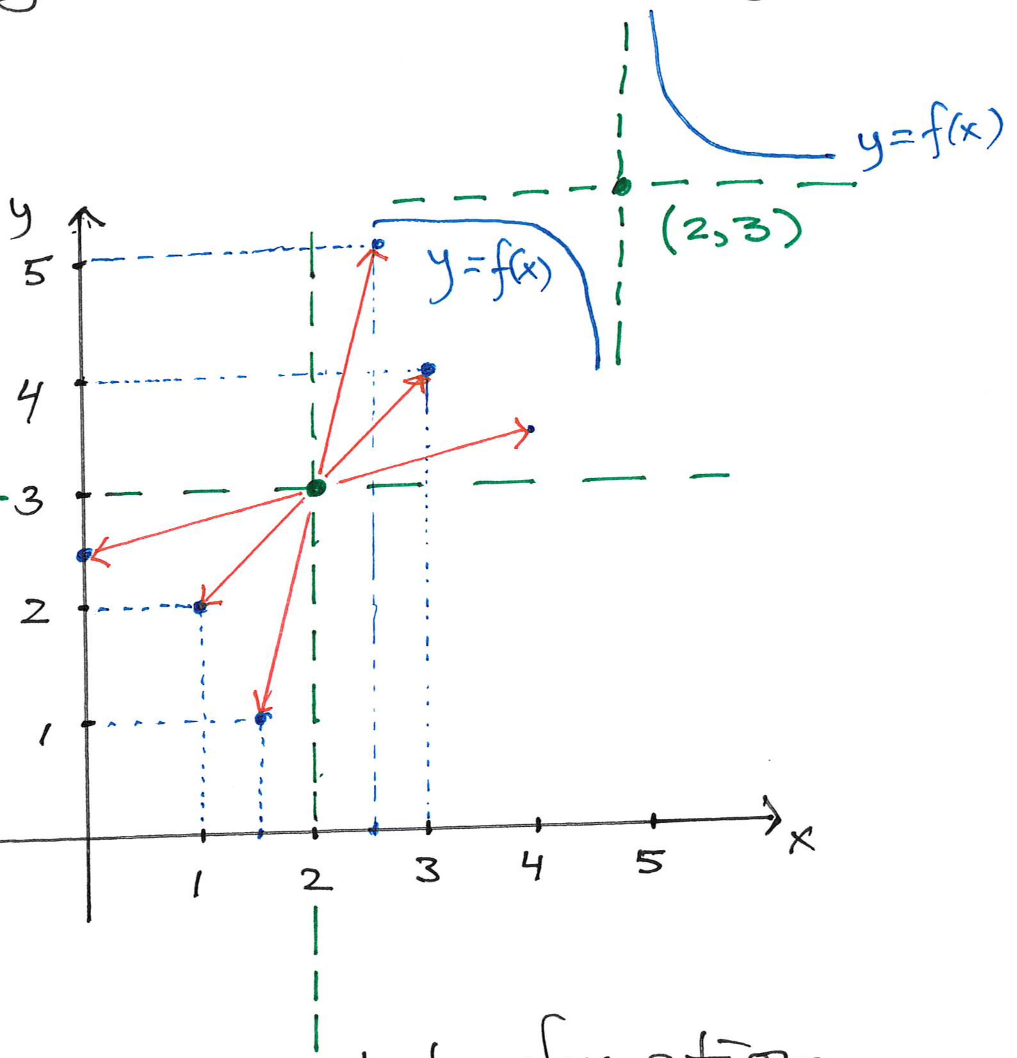
$f(3) = 3 + \frac{1}{3-2} = 4$

$f(1.5) = 3 + \frac{1}{1.5-2} = 1$

$f(2.5) = 3 + \frac{1}{2.5-2} = 5$

$f(0) = 3 + \frac{1}{0-2} = 2.5$

$f(4) = 3 + \frac{1}{4-2} = 3.5$



The graph of a hyperbola function is symmetric through the intersection point of the asymptotes.

2019s Multiple Choice

Problem 5

We have the hyperbola function $f(x) = \frac{4x - 38}{x - 10}$. Which of the graphs in figure 1 is the graph of $f(x)$?

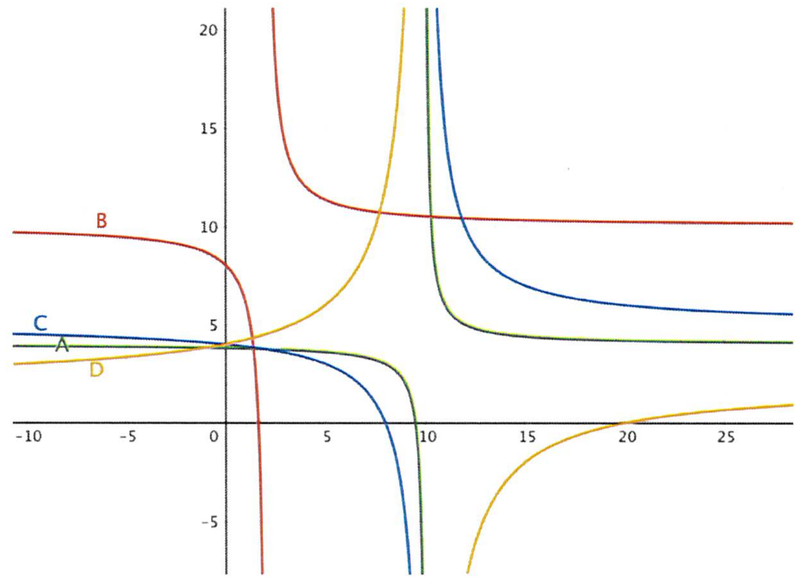


Figure 1: Graphs A-D

(A) $f(x)$ has the graph A (green)
(B) $f(x)$ has the graph B (red)
(C) $f(x)$ has the graph C (blue)
(D) $f(x)$ has the graph D (yellow)
(E) I choose not to answer this problem.

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2019 a Term Paper

Find the expression for the hyperbola function.

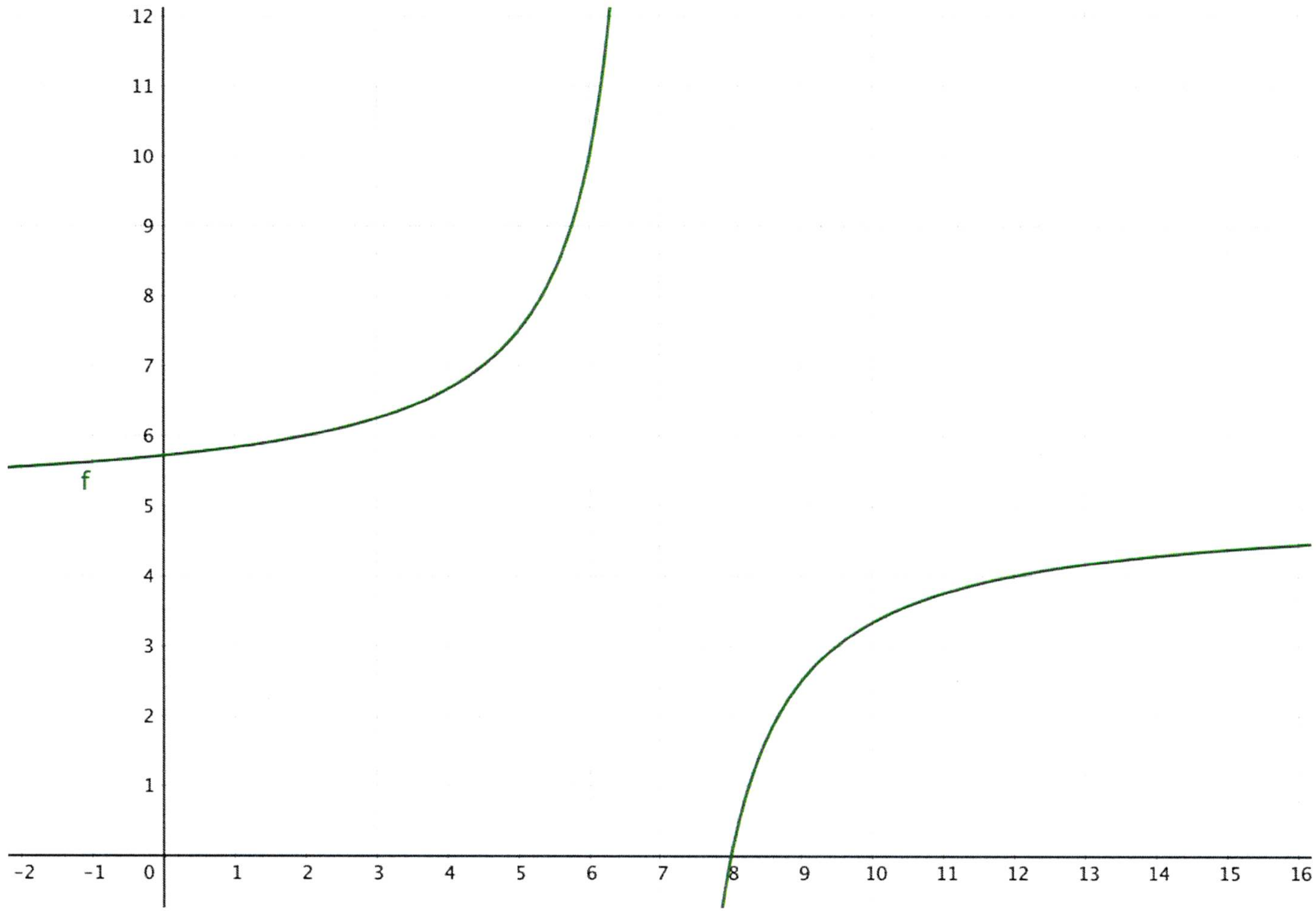

Figure 2: Hyperbola

5c