Exercise session problems

Problem 1.
Consider the vectors given by

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

Draw these vectors in a two-dimensional coordinate system. Then compute the following vectors, and draw them into the same coordinate system:

a) $\mathbf{u} + \mathbf{v}$ b) $\mathbf{v} + \mathbf{w}$ c) $\mathbf{v} - \mathbf{w}$ d) $2\mathbf{u}$ e) $-\mathbf{v}$ f) $3\mathbf{u} + \mathbf{w}$

Problem 2.
Solve the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ for the vectors below. Is $\mathbf{b}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ -8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Problem 3.
Write the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ in matrix form, and use this to solve the equation:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Problem 4.
Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Problem 5.
Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ 1 & -2 \\ 7 & 1 \end{pmatrix}$$

Compute the following expressions, whenever possible:

a) $A + B$ b) $2A - 3B$ c) $A - C$ d) AB e) BC f) ABC
g) AC h) A^2 i) BA j) CB k) C^2 l) $C^T A$

Problem 6.

Determine all (a,b,c,d) such that the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ given below. Use this to determine whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ when $(a,b,c,d) = (0,0,1,1)$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Problem 7.

You have 400.000 NOK and are to invest in a portfolio of risky assets. You can chose a combination of the stocks A, B, C with prices $p_A = 60$ NOK, $p_B = 75$ NOK and $p_C = 320$ NOK per stock at the time of investment. We assume that at a given time in the future, one of three scenarios will happen. The prices of the stocks in these scenarios are given in the table below. Denote by x, y, z the number of stocks you buy of each of the three risky

	Price A	Price B	Price C
Purchase price	60	75	320
Scenario 1	80	80	350
Scenario 2	100	25	500
Scenario 3	40	100	55

assets.For simplicity, we assume that x,y,z can be arbitrary real numbers. Hence, we allow for buying a negative number of stocks (short selling), and the number of stocks does not have to be an integer.

a. Assume that the condition $60x + 75y + 320z = 400.000$ is satisfied. What does this condition mean?

b. Denote by R_1, R_2 and R_3 the profit from the portfolio in the three scenarios. Is it possible to chose the portfolio such that $(R_1, R_2, R_3) = (50.000, 25.000, -100.000)$? Which portfolio must we chose in order for this to hold?

c. Is it possible to chose a portfolio of risky assets such that $R_1 > 0$ and $R_2 = R_3 = 0$? Which portfolio must we chose in this case? Give an interpretation of your answer.

d. Describe all (R_1,R_2,R_3) of possible profits in the three scenarios. Are there any portfolios such that $R_1, R_2, R_3 > 0$ (i.e., guaranteed profit in all scenarios)?

Problem 8.

Let A be a 2×3-matrix.

a) Is A symmetric?

b) Is $A^T A$ symmetric?

c) Compute $A^T A$ when $A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$.

Problem 9.

Solve the matrix equation for X when $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$:

a) $AX = I$

b) $X^2 = A$

c) $AX = XA$

Optional: Exercises from the Norwegian textbook.

Textbook [E]:	Eriksen, *Matematikk for økonomi og finans*
Exercise book [O]:	Eriksen, *Matematikk for økonomi og finans - Oppgaver og Løsningsforslag*

Exercises:	[E] 6.5.1 - 6.5.6
Solution manual:	See [O] Kap 6.5

Answers to exercise session problems

Problem 1.

a) $\begin{pmatrix}4\\3\end{pmatrix}$ b) $\begin{pmatrix}2\\6\end{pmatrix}$ c) $\begin{pmatrix}4\\-4\end{pmatrix}$ d) $\begin{pmatrix}2\\4\end{pmatrix}$ e) $\begin{pmatrix}-3\\-1\end{pmatrix}$ f) $\begin{pmatrix}2\\11\end{pmatrix}$

Problem 2.
The general solution is $(x,y,z) = (-4z-1, z+1, z)$ with z free. A specific solution is found by (for instance) letting $z = 0$, which gives $(-1,1,0)$. This means that $\mathbf{b} = -1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 3.
For $a = -8$ there are infinitely many solutions $(x,y,z) = (-4z-1, z+1, z)$ with z free (like in the previous exercise). For $a \neq -8$ there is exactly one solution $(x,y,z) = (-1,1,0)$.

Problem 4.
Exactly one solution $(x,y,z) = (-3/2, 4, -1/2)$.

Problem 5.

a) $\begin{pmatrix}1&2&3\\3&1&3\\0&3&5\end{pmatrix}$ b) $\begin{pmatrix}2&-1&-4\\1&2&11\\-5&-4&-10\end{pmatrix}$ c) not defined d) $\begin{pmatrix}2&3&5\\5&10&19\\2&1&1\end{pmatrix}$

e) $\begin{pmatrix}15&0\\-4&3\\33&4\end{pmatrix}$ f) $\begin{pmatrix}44&7\\158&19\\14&7\end{pmatrix}$ g) $\begin{pmatrix}11&3\\35&10\\5&-5\end{pmatrix}$ h) $\begin{pmatrix}2&3&6\\0&7&10\\0&1&4\end{pmatrix}$

i) $\begin{pmatrix}0&3&6\\2&0&0\\1&7&13\end{pmatrix}$ j) not defined k) not defined l) $\begin{pmatrix}-2&11&14\\-1&3&-3\end{pmatrix}$

Problem 6.
This is a linear combination if $-7a + 9b - 5c + 3d = 0$, and $(a,b,c,d) = (0,0,1,1)$ does not correspond to a linear combination since these values do not satisfy the equation.

Problem 7.

a. This is a budget constraint, i.e., that the total cost of the risky assets we purchase is 400.000 NOK.

b. Yes, if we choose the portfolio $(x,y,z) = (1187\,{}^1\!/\!{}_2, 2250, 500)$.

c. Yes, if $R_1 = 80.000$. Then, we have to choose the portfolio $(x,y,z) \approx (3333\,{}^1\!/\!{}_3, 2666\,{}^2\!/\!{}_3, 0)$. This means that we can invest without the risk of loss, and with positive expected profit (a very attractive situation!).

d. The triplets of profit (R_1,R_2,R_3) that are attainable satisfy the equation $5R_1 - 2R_2 - 2R_3 = 400.000$. We can choose the portfolio such that $R_1, R_2, R_3 > 0$ (i.e., guaranteen profit in all scenarios, an even more attractive situation!). For instance, we can attain the profit $R_1 = R_2 = R_3 = 400.000$.

Problem 8.

a) No

b) Yes

c) $\begin{pmatrix} 10 & 8 & 6 \\ 8 & 10 & 0 \\ 6 & 0 & 10 \end{pmatrix}$

Problem 9.

a) $X = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$

b) no solution

c) $X = s\begin{pmatrix} -4 & 3 \\ 2 & 0 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$