EBA1180, lecture 5, 7 Sept. 2022, Runar Ile

Plan: 1. Regular cash flows
2. Infinite series and limit values
3. Euler's number and continous compounding

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1. Regular cash flows

A fixed amount is paid every period/term.

Ex Annuity loan (tot. pres. value = what you can borrow)

Ex Saving with a fixed amount each period. Future value = the balance, what you have saved

– both gives geometric series.

Ex (Term paper 2019a, prob. 6a)

Kåre considers a mortgage with

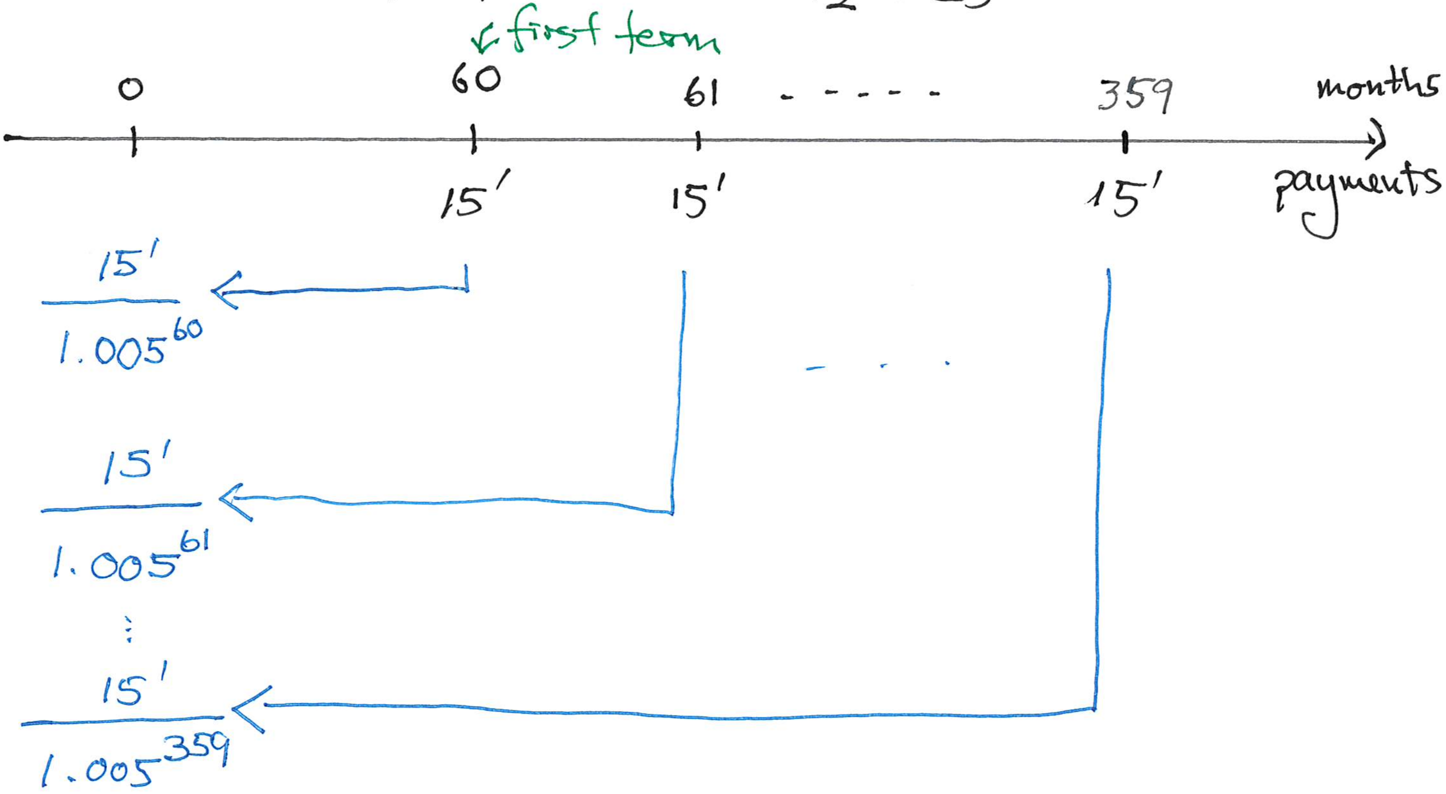
- monthly payments running for 25 years
- first payment 5 years from now
- the interest is 6 %
- Kåre reckons he can pay 15000 each month

- Determine the geom. series which gives the tot. pres. ~~value of~~ the cash flow
- Calculate how much Kåre can borrow.

Solution Kåre can borrow the tot. pres. val. of the cash flow.

The period rate is $\frac{6\%}{12} = 0.5\%$

Number of periods: $12 \cdot 25 = 300$



The sum (the tot. pres. value) is a geom. series with $a_1 = \frac{15'}{1.005^{359}}$, $n = 300$, $k = 1.005$

The tot. pres. value (what Kåre can borrow) is

$$\frac{15000}{1.005^{359}} \cdot \frac{1.005^{300} - 1}{0.005} = \underline{\underline{1734620.76}}$$

Start: 10.55

2. Infinite series and limit values

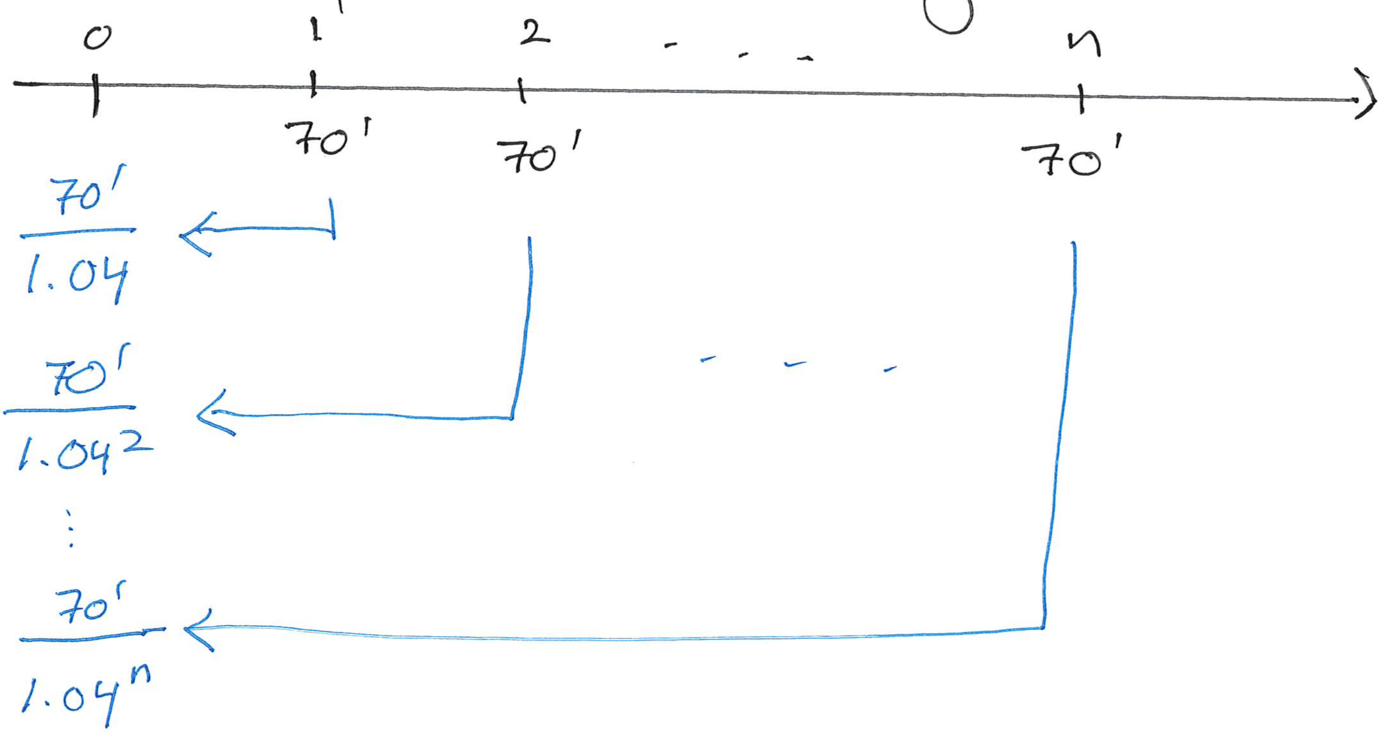
Ex The annuity : 70 000

interest : 4 %

number of years : n

First payment : One year from now.

The tot. pres. value (what you can borrow)



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the sum is a geom. series with

$$a_1 = \frac{70'}{1.04^n}, \quad n \text{ terms}, \quad k = 1.04$$

The sum is

$$\frac{70'}{1.04^n} \cdot \frac{1.04^n - 1}{0.04} = \frac{70' \cdot (1.04^n - 1)}{1.04^n \cdot 0.04}$$

$$= \frac{70' \cdot (1.04^n - 1) : 1.04^n}{\cancel{1.04^n} \cdot 0.04 : \cancel{1.04^n}} = \frac{70' \cdot \left(\frac{1.04^n}{1.04^n} - \frac{1}{1.04^n} \right)}{0.04}$$

$$= \frac{70' \cdot \left(1 - \frac{1}{1.04^n}\right)}{0.04}$$

approaches 0
when $n \to \infty$
"n goes to infinity"

So the tot. pres.
value is
approaching

$$\frac{70'}{0.04} \quad \text{when} \quad n \to \infty$$

$$= \underline{\underline{1\,750\,000}}$$

<u>Conclusion</u> If you pay the bank 70 000 each year, starting next year, and the interest i 4 %, and you pay <u>forever</u>, then you can borrow 1.75 mill.

3. Euler's number and continuous compounding

Ex You deposit 1000 into an account with 12% nominal interest.

compounding	balance after 1 year
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quarterly	$1000 \cdot 1.03^4 = 1125.51$
monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1 + \frac{12\%}{365}\right)^{365} = 1127.47$

Pattern (n periods) $\quad 1000 \cdot \left(1 + \frac{0.12}{n}\right)^n$

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Euler's number: $\quad e = 2.718281\ldots$

Calculator: $1 \ \boxed{e^x}$

Calculate: $1000 \cdot e^{0.12} = 1127.50$

$1000 \ \boxed{\times} \ 0.12 \ \boxed{e^x} \ \boxed{=}$

Euler's number e is defined as the limit of $\left(1+\frac{1}{n}\right)^n$ when $n \to \infty$

Write $\quad \left(1+\frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$

Ex $\left(1+\frac{1}{1000}\right)^{1000} = 2.71692\ldots$

$$\left(1+\frac{1}{1\text{mill}}\right)^{1\text{mill}} = 2.718280\ldots$$

Back to the example with 12 %

$$\left(1+\frac{0.12}{n}\right)^{n} = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{n}$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\text{approaches } e \text{ as } n\to\infty}\Bigg]^{0.12} \xrightarrow[n\to\infty]{} e^{0.12}$$

So $1000\cdot\left(1+\frac{0.12}{n}\right)^{n} \longrightarrow 1000\cdot e^{0.12}$

After 1 year with 12 % nominal interest and continuous compounding the deposit of 1000 has increased to

$$1000\cdot \boxed{e^{0.12}} = 1127.50$$

(the growth factor for 1 year with continuous compounding

Annual growth factor: $e^{0.12} = 1.12750$

The <u>effective</u> interest $e^{0.12} - 1 = 0.12750$

$= 12.75\%$

After 2 years of continuous compounding

$$1000 \cdot e^{0.12} \cdot e^{0.12} = 1000 \cdot e^{0.12 + 0.12}$$

$$= 1000 \cdot e^{0.12 \cdot 2}$$

$$= 1000 \cdot e^{0.24}$$

$$= \underline{\underline{1271.25}}$$