Review EBA1180 Spring semester

EBA1180
Exam
review
Spring 23

3 main topics:

1) Integration
2) Linear algebra
3) Functions in two variables

1: Integration

- Integration as antidifferentiation:

$$\int f(x)\,dx = \underbrace{F(x) + C}_{\text{all antiderivatives of } f(x)} \quad \text{where } F'(x) = f(x)$$

- Integration methods:

→ Substitution: integration version of chain rule

$$du = u'\,dx \quad u'(x)$$
$$dx = \frac{1}{u'}\,du$$

Recognize: "Ugly" inner function. Perhaps its derivative as a factor.

→ Integration by parts: integration version of the product rule

$$\int u' \cdot v\,dx = u \cdot v - \int u \cdot v'\,dx$$

Recognize: Product & something nice when antidifferentiated!

Integration of rational functions:

i) $\int \frac{2}{1-x}\,dx$ ii) $\int \frac{2x}{1-x^2}\,dx$

iii) $\int \frac{2}{1-x^2}\,dx$

To solve: i) Substitute $u = 1-x$.

ii) Substitute $u = 1-x^2$.

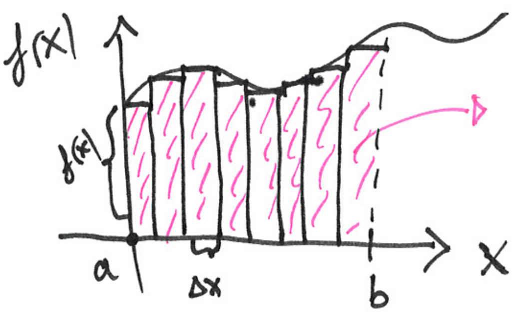
iii) Partial fractions:

$$\underbrace{\frac{2}{1-x^2}}_{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x},$$ solve for A, B.

Then, solve integral as type i).

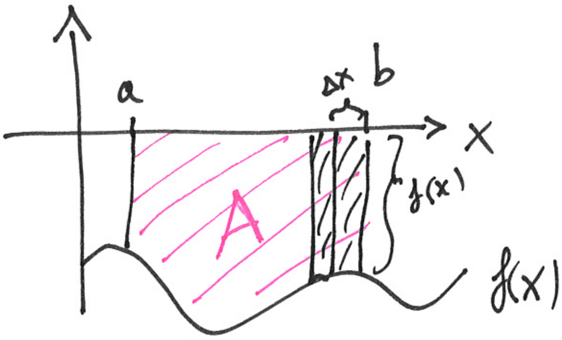
- Definite integrals: $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F'(x) = f(x)$.



sum of area of rectangles $\longrightarrow$ area under graph of f between a and b $= \int_a^b f(x)\,dx$

Δx width $\to 0$



$$A = (-) \int_a^b f(x)\,dx$$

OBS!

- **Economic applications:**

Total cash flow: $\int_0^T f(x)\,dx$ — $f(x)$: cash flow per time unit

NPV of cash flow with continuous discounting: $\int_0^T f(x)\,\underbrace{e^{-rx}}\,dx$ — r: discount rate

2 : Linear algebra

→ matrices and matrix operations can be used to solve systems of linear equations:

- Gaussian elimination (elemenentary row operations)

- # solutions can be seen from the echelon form of the augmented matrix.

Thm: Any linear system has either

i) No solutions. $\rightsquigarrow$ Inconsistent

ii) One unique solution.
iii) Infinitely many solutions.
$\left.\right\}$ Consistent

$\rightarrow$ Computing with matrices:

- $A+B$
- $A-B$

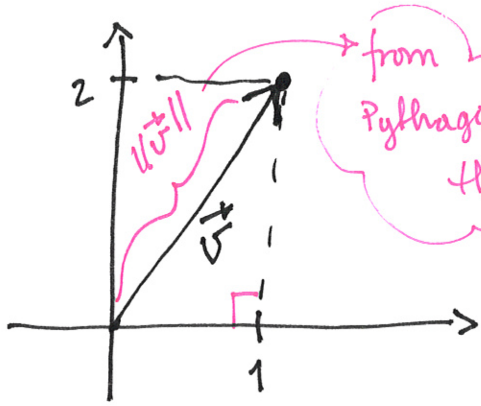
$\left.\right\}$ A, B same size

- $rA \rightarrow$ always defined
- $A \cdot B \rightarrow$ # columns in A = # rows in B

OBS: $A \cdot B \neq B \cdot A$

$\rightarrow$ Length of vectors: $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



from Pythagorean thm.

Can visualize $\vec{v}+\vec{w}$, $\vec{v}-\vec{w}$, $r\vec{v}$ etc. in the figure.

$\rightarrow$ Determinants:

$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$; For square matrices.

$|A| = \det(A)$

- $\underline{2\times 2}$: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- $\underline{n\times n, \ n>2}$: Cofactor expansion along row / column. → Choose wisely! 0's make calculations easier.

- Computations: $|A\cdot B| = |A|\cdot|B|$

  $|c\cdot A| = c^n |A|$, for A $n\times n$

  $|A^T| = |A|$

Result: i) $|A| \neq 0 \Rightarrow$ One unique solution to the corresponding linear system.

ii) $|A| = 0 \Rightarrow$ No solutions or infinitely many solutions.

- Cramer's rule: Explicit solution of square linear system with $|A| \neq 0$.

→ Linear combinations: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

→ Transpose: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$

"A flipped"

Calculations: $(A^T)^T = A$

$$(AB)^T = B^T A^T$$

→ Inverse matrices: For square matrices.

A^{-1} is the matrix s.t.

$$A^{-1} \cdot A = I \text{ and}$$
$$A \cdot A^{-1} = I$$

• 2×2: $|A| \neq 0$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = 0$: No inverse.

• n×n: $|A| \neq 0$: $\quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$

where C_{ij} are the cofactors.

↳ ALT: $[A \mid I] \sim \ldots \sim [I \mid A^{-1}]$

$|A| = 0$: No inverse.

- Solve equations via the inverse: If A^{-1} exists

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

→ Inner product/ dot product of vectors:

same length

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, then

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

oo

3: Functions in two variables

A: Optimization without constraints

$$\max / \min \ f(x, y)$$

i) Find candidates for max/min:

I) Stationary points: $f'_x = 0$ and $f'_y = 0$

II) Points where f'_x or f'_y are not defined.

III) Boundary points.

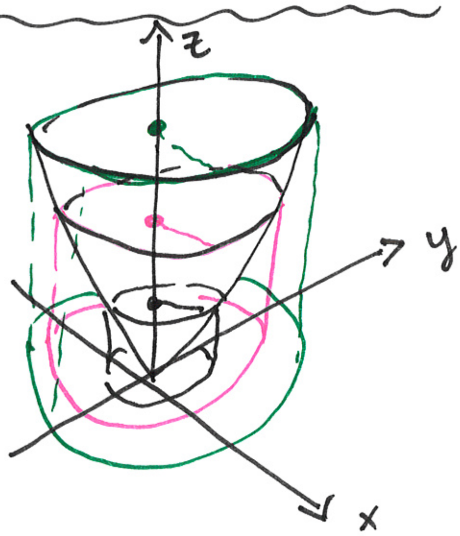
ii) Classify candidates as local max / local min / saddle point:

Second derivative test.

iii) Analyze to see whether local max / local min are (global) max / min.

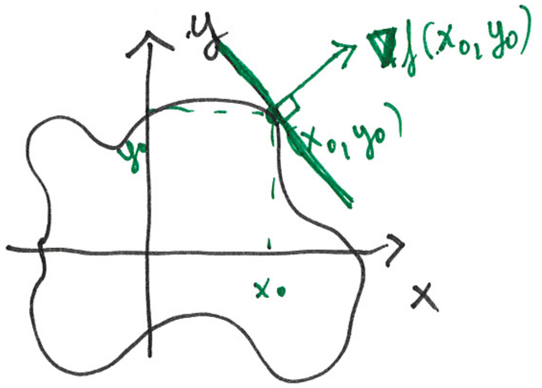
Can you find a ~~smaller~~ bigger value for f than the local ~~min~~/max → Not a global max/min

B: Level curves: $f(x, y) = c$



Level curves are curves in the xy-plane.

$z = f(x, y)$ floats above.



i) Tangent to $f(x, y) = c$ in (x_0, y_0):

Point-slope formula + implicit differentiation

$$y - y_0 = a(x - x_0) \text{ where}$$

$$a = -\frac{f'_x(x_0, y_0)}{f'_y(x_0, y_0)}$$

ii) Gradients: $\nabla f = \begin{bmatrix} f'_x \\ f'_y \end{bmatrix}$ is a vector which is normal on the tangent and points in the direction where f grows the fastest.

C: Optimization with constraints

max/min $f(x,y)$ when ....

Constraints: Equality or inequality

- D = set of admissible points (i.e. those points satisfying all of the constraints)

Think of this as a set in xy-plane

Extreme value theorem: f continuous and D compact (closed and bounded) $\Rightarrow$

f has a max and min on D.

Different cases:

I) Optimization on a rectangle:

$$\boxed{\begin{array}{c} a \le x \le b \\ c \le y \le d \end{array}}$$

Candidates for max/min:

i) Stationary points that are inner points for D ($a < x < b$, $c < y < d$) or other inner critical points $\Rightarrow$ (partial derivatives not def.)

ii) Boundary points for D: Analyze separately.

Max / min: Candidate point with largest/ smallest f-value.

If f is continuous

$\Downarrow$

EVT holds

II) Lagrange problems: max / min $f(x,y)$ when $g(x,y) = a$

Candidates for max/min:

i) Lagrangian: $\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\,(g(x,y) - a)$

Lagrange conditions:

FOC: $\mathcal{L}'_x = 0$

$\mathcal{L}'_y = 0$

C: $g(x,y) = a$

$\Rightarrow (x,y;\lambda)$: Ordinary candidate point

ii) Admissible points with degenerate constraint:

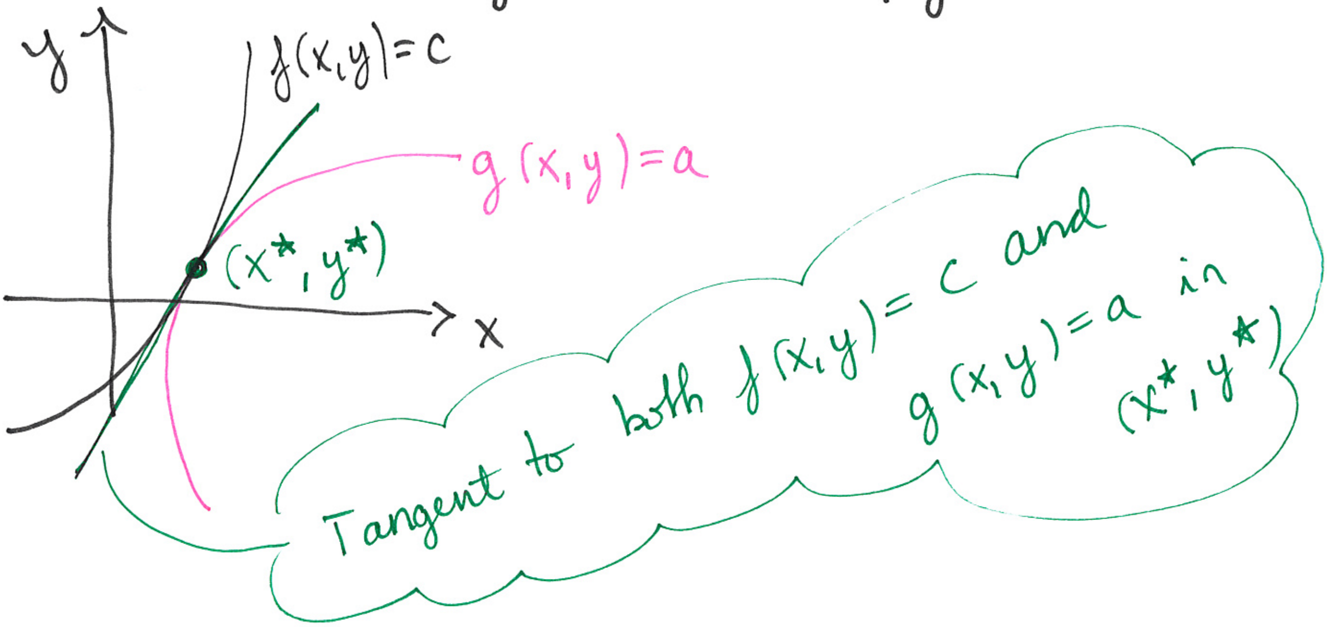
$$\boxed{\begin{aligned} g'_x &= 0 \\ g'_y &= 0 \\ g(x,y) &= a \end{aligned}} \Rightarrow (x,y) \text{ candidate}$$

NOTE: A point $(x^*, y^*; \lambda^*)$ satisfies the Lagrange condition FOC + C with $f(x^*, y^*) = c$

a)

$$\Updownarrow$$

The level curve $f(x,y) = c$ and the constraint $g(x,y) = a$ have the same tangent in (x^*, y^*).



b) An point (x^*, y^*) with degenerate constraint

admissible

$$\Updownarrow$$

The curve $g(x,y) = a$ does not have a unique tangent in (x^*, y^*).

c) Interpretation of λ: $(x^*, y^*; \lambda^*)$ is max/min $\Rightarrow$

λ^* = marginal change in the max/min value when we change a in the constraint $g(x,y) = a$

III) Kuhn-Tucker problems:

$$\max/\min f(x,y) \text{ when } g(x,y) \leq a$$

Method: Split into Lagrange problem $(=)$ and analysis of inner points $(<)$.

inner stationary or other critical points