(Look at Lagrange problem part of the Kuhn Tucker problem)

Ex: $\max f(x,y) = x^2y^2$ when $x^2+y^2+x^2y^2 = 3$

$\mathcal{L} = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

CANDIDATE POINTS:

2 kinds

i) Stationary pts. of $\mathcal{L}$

ii) Degenerate constraint

(From Theorem)

i) Stationary points of $\mathcal{L}$:

$\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0$ (1)

$\mathcal{L}'_y = x^2 \cdot 2y - \lambda(2y + x^2 2y) = 0$ (2):

C: $\mathcal{L}'_\lambda = x^2+y^2+x^2y^2 = 3$ (3):

From (1):

$2x(y^2 - \lambda - \lambda y^2) = 0$ → $x = 0$ OR $y^2 - \lambda - \lambda y^2 = 0$

From (2):

$2y(x^2 - \lambda - \lambda x^2) = 0$ → $y = 0$ OR $x^2 - \lambda - \lambda x^2 = 0$

(Check all combinations:) ⇓

a) $x = 0,\ y = 0$: From (3): $0^2 + 0^2 + 0^2 0^2 = 3$; Impossible!

⇒ No candidates.

b) $x = 0,\ x^2 - \lambda - \lambda x^2 = 0$: → $0^2 - \lambda - 0 = 0 \Rightarrow \lambda = 0$

From (3): $0^2 + y^2 + \underbrace{0 \cdot y^2}_{0} = 3 \Rightarrow y = \pm\sqrt{3}$

So: Candidates: $(0, \sqrt{3}; \underbrace{0}_{\lambda})$, $(0, -\sqrt{3}; \underbrace{0}_{\lambda})$

$f(0, \sqrt{3}) = 0$ $\qquad$ $f(0, -\sqrt{3}) = 0$

(c) $y = 0,\ y^2 - \lambda - \lambda y^2 = 0$:

$0^2 - \lambda - \lambda \cdot 0^2 = 0 \Rightarrow \lambda = 0$

From (3):

$x^2 + 0^2 + x^2 0^2 = 3 \Rightarrow x^2 = 3$, so $x = \pm\sqrt{3}$

Candidates: $(0, \sqrt{3}; 0)$, $(0, -\sqrt{3}; 0)$

d) $y^2 - \lambda - \lambda y^2 = 0,\ x^2 - \lambda - \lambda x^2 = 0$:

$y^2 = \lambda(1 + y^2) \Rightarrow \lambda = \frac{y^2}{(1+y^2)} \rightarrow$ never 0

$x^2 = \lambda(1 + x^2) \Rightarrow \lambda = \frac{x^2}{1+x^2}$ (★) $\rightarrow$ never 0

$\Rightarrow$ $\lambda = \lambda$ $\quad \frac{y^2}{1+y^2} = \frac{x^2}{1+x^2} \quad \Big| \cdot (1+y^2)(1+x^2)$

$y^2(1 + x^2) = x^2(1 + y^2)$

$y^2 + \cancel{y^2 x^2} = x^2 + \cancel{x^2 y^2}$

$y^2 = x^2$

$y = \pm x$: (★)

From (3):

$$x^2 + x^2 + x^2 x^2 = 3$$

$$x^4 + 2x^2 - 3 = 0, \quad \text{let } u = x^2:$$

$$u^2 + 2u - 3 = 0 \text{ ; a \underline{quadratic eqn}:}$$

$$u = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{16}}{2} = \frac{-2 \pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

So $x^2 = 1$ or $\cancel{x^2 = -3}$

NOT POSSIBLE!

$$x = \pm 1$$

Hence, $y = \pm x = \pm 1$ → OBS! 4 combinations:

From $(\tilde{*})$

$x = 1$: $y = 1$, $y = -1$

$x = -1$: $y = -1$, $y = 1$

Also, $\lambda = \frac{x^2}{1 + x^2} = \frac{1}{2} \Rightarrow$

From (*)

Candidates: $(1, 1; \frac{1}{2}), (1, -1; \frac{1}{2}),$

$(-1, 1; \frac{1}{2}), (-1, -1; \frac{1}{2})$ $\Big\}$ (*)

For all of these: $f = 1$

$x^2 + y^2$
(sign irrelevant due to square)

ii) Admissible points with degenerate constraints:

$$g(x, y) = x^2 + y^2 + x^2 y^2 = 3$$

$$g'_x = 2x + 2xy^2 = 0 \Rightarrow 2x\underbrace{(1+y^2)}_{\text{Never } 0} = 0 \Rightarrow x = 0$$

$$g'_y = 2y + x^2 2y = 0 \Rightarrow 2y\underbrace{(1+x^2)}_{\text{Never } 0} = 0 \Rightarrow y = 0$$

$\Rightarrow (x, y) = (0, 0)$; But this is not admissible since then

$$g(0,0) = 0^2 + 0^2 + 0^2 0^2 = 0 \neq 3,$$ so

the constraint doesn't hold.

Hence, no candidates of type ii)

Is there a max?

$D: x^2 + y^2 + x^2 y^2 = 3$

• Closed? ✓

• Bounded? ✓

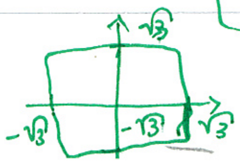
Note that x^2, y^2 and $x^2 y^2 \geq 0$ $\forall$ x, y. Hence, both x^2 and y^2 must be ≤ 3. (WHY?) But then:

$x^2 \leq 3$ and $y^2 \leq 3 \iff$

$-\sqrt{3} \leq x \leq \sqrt{3}$

$-\sqrt{3} \leq y \leq \sqrt{3}$

The admissible x's and y's fit into this ~~rectangle~~ square:



Conclusion: $f_{max} = 1$ at $(1,1), (1,-1), (-1,1)$ and $(-1,-1)$ with $\lambda = \frac{1}{2}$

So D is bounded $\Rightarrow$ D is compact & f cont. $\Rightarrow$ EVT holds $\Rightarrow$ $\exists$ max and min.

(Briefly mention $\leq$ or $\geq$ constraints:)

Kuhn-Tucker problems

→ Optimization problems with closed inequality constraints ($\leq$, $\geq$).

Ex: $\boxed{\max\ f(x,y) = x^2y^2 \text{ when } x^2+y^2+x^2y^2 \leq 3}$

A Kuhn-Tucker problem

Candidate points:

(1) Boundary points: Boundary = constraint!

⇒ Lagrange problem.

Solve this by std. Lagrange →
(Will do this at end of lecture (LONG!))

(2) Interior: Stationary or other critical interior points of f.

Strict since interior point

Ex: $f'_x = 2xy^2 = 0 \Rightarrow x=0$ or $y=0$

$f'_y = x^2 2y = 0 \Rightarrow x=0$ or $y=0$

NOTE: These are always defined (no division by 0 etc) ⇒ No other interior critical points

Candidates: • $(0,y)$ when $0^2+y^2+0^2y^2 < 3$

$y^2 < 3$

$-\sqrt{3} < y < \sqrt{3}$

$f(0,y) = 0^2 \cdot y^2 = 0$

- $(x, 0)$ when $x^2 + 0^2 + x^2 0^2 < 3$

interior point, so strict ineq.

$x^2 < 3$

$-\sqrt{3} < x < \sqrt{3}$

$f(x,0) = x^2 0^2$
$= 0$

→ From same argument as before $D: x^2+y^2+x^2y^2 \leq 3$ is compact (it is closed since $\leq$ and see prev. ex. for why it is bounded). Also, f is continuous, so the EVT holds.

$\Downarrow$

There is a max (and a min) of the Kuhn-Tucker problem.

CONCLUSION: By comparing function values of the candidates,

$f_{max} = 1$ (at the same points as before, see (*)).