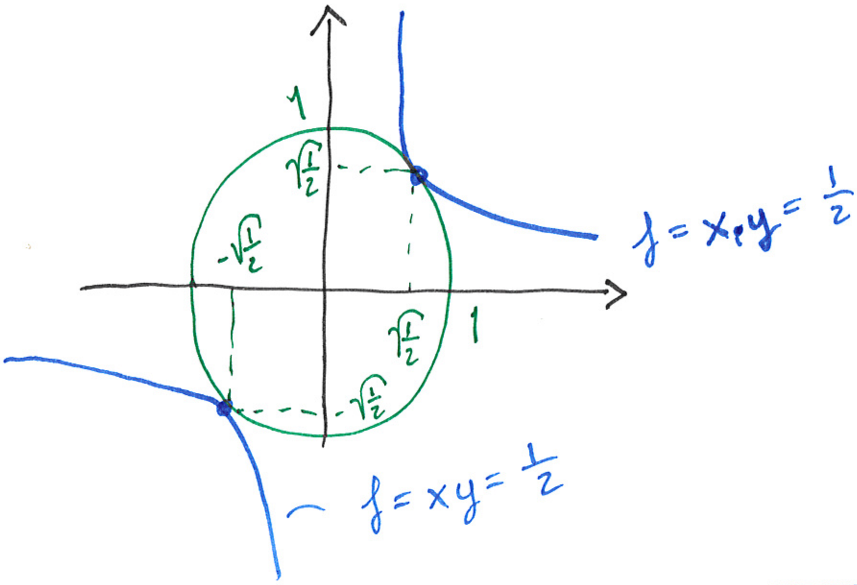
Interpretation of Lagrange multipliers

EBA 1180
Lecture 47
Spring 23

<u>Ex</u>: max/min $f(x,y) = xy$ when $\underbrace{x^2+y^2}_{g(x,y)} = 1$ — circle, center $(0,0)$, $r=1$

$f_{max} = \frac{1}{2}$ at

$\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$, $\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ with $\lambda = \frac{1}{2}$.



<u>Def</u>: Consider max $f(x,y) = xy$ with $x^2+y^2 = a$.

<u>Max. point:</u> $(x^*(a), y^*(a))$.

<u>Max. value:</u> $f(x^*(a), y^*(a)) = f^*(a)$

<u>Ex</u>: <u>a=1</u>: $x^*(1) = \sqrt{\tfrac{1}{2}}$, $y^*(1) = \sqrt{\tfrac{1}{2}}$, $f^*(1) = \frac{1}{2}$

OR

$x^*(1) = -\sqrt{\tfrac{1}{2}}$, $y^*(1) = -\sqrt{\tfrac{1}{2}}$, $f^*(1) = \frac{1}{2}$

Result:

$$\lambda = \frac{d f^*(a)}{d a}$$

Lagrange multiplier

Change in max value when a changes

small change in parameter a

Interpretation of λ: λ is the marginal change in the max (min) value per unit change in the constant a in the constraint $g(x,y) = a$.

Ex: USE THIS TO APPROXIMATE OPTIMAL VALUES :

$a=2$:

$$f^*(2) \approx f^*(1) + \Delta a \frac{d f^*(a)}{d a}$$

$$= \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= \underline{1}$$

Example + Result

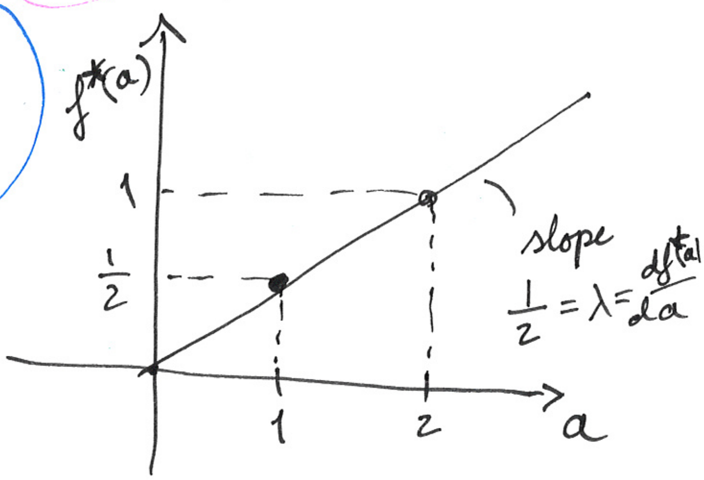
Def. of derivative:

$$f'(a) \approx \frac{\Delta f(a)}{\Delta a}$$

Actually: $f'(a) = \lim_{\Delta a \to 0} \frac{\Delta f(a)}{\Delta a}$

$$\Delta f^*(a) \approx \Delta a \, (f^*)'(a)$$

$$f^*(2) - f^*(1) \approx (2-1) \frac{d f^*(a)}{d a}$$



Ex, ctd: max $f(x,y) = xy$ when $\underbrace{x^2+y^2=a}$

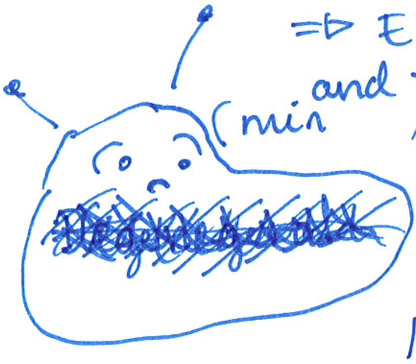
Circle, center $(0,0)$, $r = \sqrt{a}$

EVT? • $f(x,y)$ continuous? Yes.

• Compact constraint set? → Closed? Yes.

→ Bounded? Yes.

⇒ EVT holds! The problem has a max (and min).



(Type ii points)

Degenerate constraint in admissible point?

No, since the constraint is a circle (see prev. lecture).

$\lambda \quad \gamma$

Type (i) points: $\mathcal{L}(x,y) = xy - \lambda(x^2+y^2-a)$

FOC:

(1): $\mathcal{L}'_x = y - \lambda \cdot 2x = 0 \Rightarrow y = 2\lambda x$

(2): $\mathcal{L}'_y = x - \lambda \cdot 2y = 0 \Rightarrow x - \lambda \cdot 2 \cdot 2\lambda x = 0$

C: c: $x^2 + y^2 = a$

$x - 4\lambda^2 x = 0$

$x(1 - 4\lambda^2) = 0$

$x = 0$:

$\lambda^2 = \frac{1}{4}$:

So: 3 cases

$x = 0$:

From (1): $y = 2 \cdot \lambda \cdot 0 = 0$

From c: $0^2 + 0^2 = a$

$0 = a$

For $a = 0$:

$(0, 0; \lambda)$ ↳ can be anything

$f = 0$

If $a \neq 0$: No candidate point.

$\lambda = \frac{1}{2}$:

From (1):

$y = 2 \cdot \frac{1}{2} \cdot x = x$

From c: $y^2 + y^2 = a$

$2y^2 = a$

$x = y = \pm\sqrt{\frac{a}{2}}$

Candidates:

$\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; \frac{1}{2}\right)$ $f = \frac{a}{2}$

and

$\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; \frac{1}{2}\right)$ $f = \frac{a}{2}$

$\lambda = -\frac{1}{2}$:

From (1):

$y = 2 \cdot \left(-\frac{1}{2}\right)x = -x$

From c:

$y^2 + y^2 = a$

$2y^2 = a$

$y = \pm\sqrt{\frac{a}{2}}$

Candidates:

$\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; -\frac{1}{2}\right)$ $f = -\frac{a}{2}$

and

$\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; -\frac{1}{2}\right)$ $f = -\frac{a}{2}$

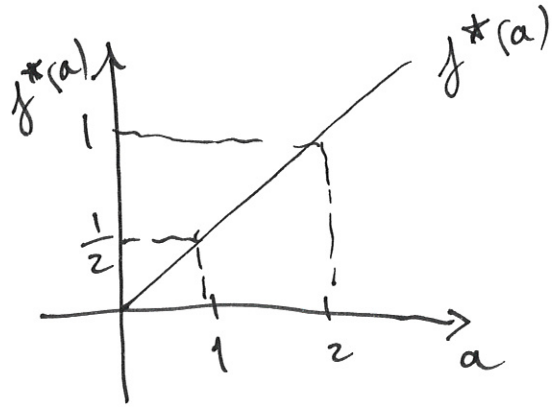
Conclusion: $f_{max} = \frac{a}{2}$ at the max. points

$\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$ and $\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$.

$f^*(a) = \frac{a}{2} = \frac{1}{2}a$

λ

$$\boxed{\frac{d f^*(a)}{d a} = \frac{1}{2} = \lambda}$$ RESULT



Kuhn - Tucker problems

→ Optimization problems with closed inequality constraints ($\leq$, $\geq$)

Ex: $\boxed{\max\ f(x,y) = x^2y^2 \text{ when } x^2+y^2+x^2y^2 \leq 3}$

Kuhn - Tucker problem

Candidate points

(1) Boundary points: Boundary ⇒ = constraint.

⇒ $\boxed{\text{Lagrange problem}}$ Solve via std. Lagrange.

(2) Interior: Stationary or other interior critical points of f.

NOTE: Always defined ⇒ No other interior critical points

Ex: $f'_x = 2xy^2 = 0 \Rightarrow x=0$ OR $y=0$

$f'_y = x^2 2y = 0 \Rightarrow x=0$ OR $y=0$

strict ineq. since interior point

Candidates: • $\overset{x=0:}{(0,y)}$ when $0^2+y^2+0^2y^2 < 3$

$y^2 < 3$

$-\sqrt{3} < y < \sqrt{3}$

$f(0,y) = 0^2y^2 = 0$

- $\underline{y=0}$:

$(x, 0)$ when $x^2 + 0^2 + x^2 0^2 < 3$

$$x^2 < 3$$

$$-\sqrt{3} < x < \sqrt{3}$$

$$f(x,0) = x^2 \cdot 0^2 = 0$$

Lagrange problem / The boundary:

$$\max \ f(x,y) = x^2 y^2 \ \text{when} \ x^2 + y^2 + x^2 y^2 = 3$$

CANDIDATES:

i) Stationary pts. of $\mathcal{L}$

ii) Degenerate constraint (From Thm)

(i) Stationary points of $\mathcal{L}$:

$$\mathcal{L} = x^2 y^2 - \lambda \left(x^2 + y^2 + x^2 y^2 - 3\right)$$

(1): $\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0$

(2): $\mathcal{L}'_y = x^2 2y - \lambda(2y + 2yx^2) = 0$

C: (3): $x^2 + y^2 + x^2 y^2 = 3$

From (1): $2x(y^2 - \lambda - \lambda y^2) = 0$ → $x = 0$ OR $y^2 - \lambda - \lambda y^2 = 0$

From (2): $2y(x^2 - \lambda - \lambda x^2) = 0$ → $y = 0$ OR $x^2 - \lambda - \lambda x^2 = 0$

Check all compinations :

a) $x=0,\ y=0$:

From (3):

$$0^2+0^2+0^2 0^2 = 3 \Rightarrow$$

Impossible! No candidates.

b) $x=0,\ x^2-\lambda-\lambda x^2=0$: $\Rightarrow$ $0^2-\lambda-0=0 \Rightarrow \underline{\lambda=0}$

From (3) : $0^2+y^2+0^2y^2=3 \Rightarrow y^2=3 \Rightarrow y=\pm\sqrt{3}$

So: Candidates: $(0,\sqrt{3};0),\ (0,-\sqrt{3};0)$

$f(0,\sqrt{3})=0$ $\quad$ $f(0,-\sqrt{3})=0$

c) $y^2-\lambda-\lambda y^2=0,\ y=0$:

See online notes.

d) $y^2-\lambda-\lambda y^2=0,\ x^2-\lambda-\lambda x^2=0$:

See online notes.

$\Rightarrow$ Candidates: $(1,1;\tfrac{1}{2}),\ (1,-1;\tfrac{1}{2}),$
$(-1,1;\tfrac{1}{2}),\ (-1,-1;\tfrac{1}{2})$

For all of these : $\underline{f=1}$

ii) <u>Admissible points with degenerate constraints:</u>

$$\left.\begin{aligned} g'_x &= 2x\underbrace{(1+y^2)}_{\text{Never } 0} = 0 \quad \Rightarrow \quad x = 0 \\ g'_y &= 2y\underbrace{(1+x^2)}_{\text{Never } 0} = 0 \quad \Rightarrow \quad y = 0 \end{aligned}\right\} \Rightarrow x = y = 0,$$

but this is not admissible:

$$g(0,0) = 0^2 + 0^2 + 0^2 0^2 = 0 \neq 3,$$

so constraint doesn't hold $\Rightarrow$

No candidates of type ii).

Conclusion: $f_{max} = 1$ at $(1,1), (1,-1),$
$(-1,1), (-1,-1).$