EBA1180
Spring 23

Warm-up:

Q: • What does the Lagrange theorem say?

• What kind of points are candidates for the max/min in a Lagrange problem?

oo

Thm: If (x^*, y^*) is max/min in a Lagrange problem:

$$\boxed{\max/\min\ f(x,y) \quad \text{with} \quad g(x,y) = a}$$

Then either

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfies the Lagrange constraints FOC + C:

FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

OR

ii) The constraint is degenerate at (x^*, y^*), i.e.,

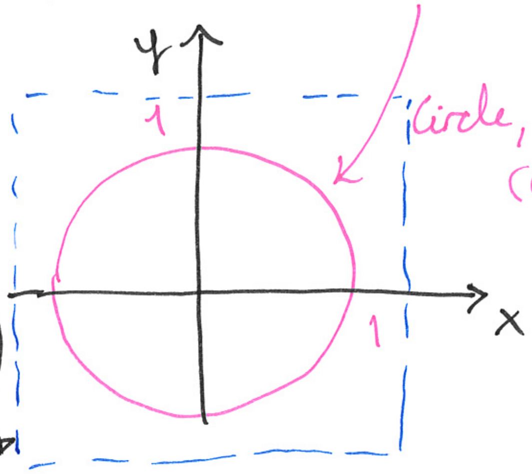
$g'_x = 0$ and $g'_y = 0$ and $g(x,y) = a$

More Lagrange problems

Ex: max/min $f(x,y) = xy$ when $\overbrace{x^2+y^2}^{g(x,y)} = 1$

→ NOTE:

$D: x^2+y^2=1$ is compact $\left(\begin{array}{l}\text{closed } \checkmark \ (=) \\ \text{bounded } \checkmark\end{array}\right)$



Circle, center $(0,0)$, $r=\sqrt{1}=1$

$\Downarrow$

From EVT: f has a max and min over D since it is continuous.

→ Degenerate constraint? $g'_x = 2x = 0 \Rightarrow x=0$

$g'_y = 2y = 0 \Rightarrow y=0$, but

then $x^2+y^2 = 0^2+0^2 = 0 \neq 1$, so not admissible.

Hence, there are no admissible points with degenerate constraint $\Rightarrow$ No type ii) candidates (ref. Thm.).

Remark: This holds in general for circles.

Lagrangian:

$$\mathcal{L} = xy - \lambda(x^2+y^2-1)$$

$$\boxed{\begin{aligned} \text{FOC:}\quad & \mathcal{L}'_x = y - \lambda \cdot 2x = 0 \\ & \mathcal{L}'_y = x - \lambda \cdot 2y = 0 \\ \text{C:}\quad & x^2 + y^2 = 1 \end{aligned}}$$

3 equations with 3 unknowns; x, y, λ

$y = 2\lambda x$

$x - \lambda \cdot 2(2\lambda x) = 0$

$x(1 - 4\lambda^2) = 0$

$x = 0$: OR $1 - 4\lambda^2 = 0$

$y = 2 \cdot \lambda x = 0$

$x^2 + y^2 = 0^2 + 0^2 = 0 \neq 1$

so C: doesn't hold.

$\Rightarrow$ Not a candidate point.

$1 - 4\lambda^2 = 0$

$\lambda^2 = \frac{1}{4}$:

$\lambda = \frac{1}{2}$:

$y = 2 \cdot \frac{1}{2} \cdot x = x$

C: $x^2 + y^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = y = \pm\sqrt{\frac{1}{2}}$

$\lambda = -\frac{1}{2}$:

$y = 2(-\frac{1}{2})x = -x$

C: $x^2 + y^2 = 1$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$y = -x$

Candidate points:

$\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$

$f(x,y) = xy = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = \frac{1}{2}$

$\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

$f = \frac{1}{2}$

MAX

Candidate points:

$\left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

$f = -\frac{1}{2}$

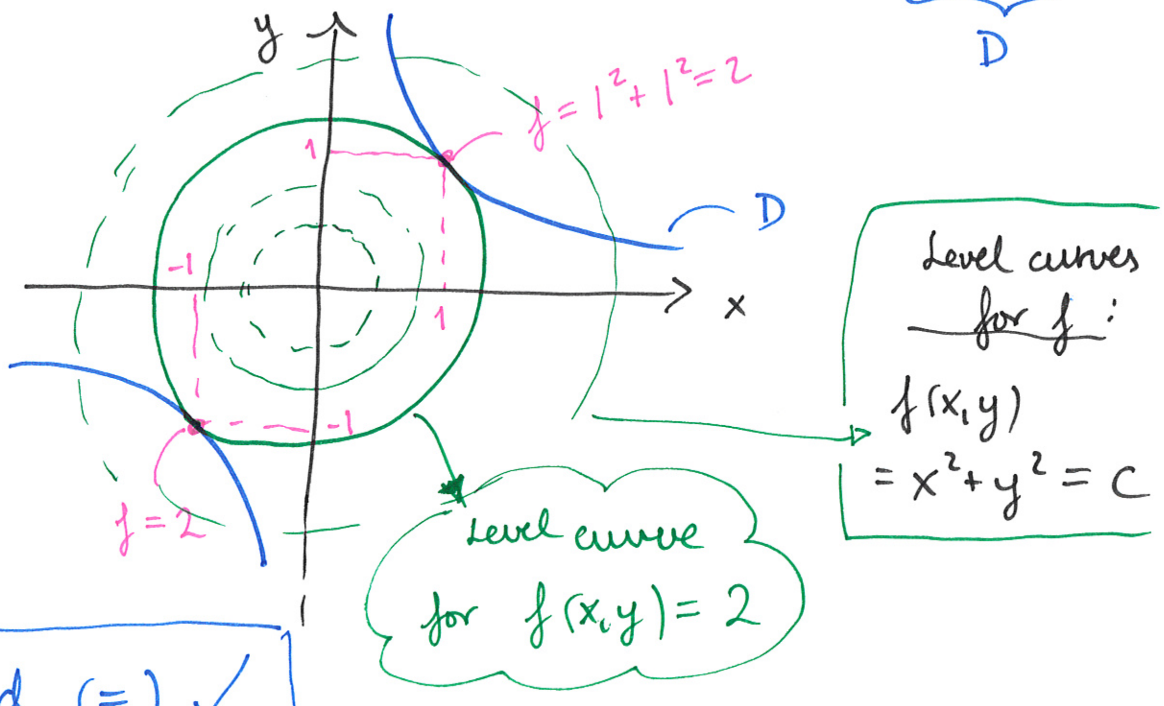
$\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$

$f = -\frac{1}{2}$

MIN

Ex: max/min $f(x,y) = x^2 + y^2$ when $\underbrace{xy = 1}_{D}$

hyperbola



EVT? Closed (=) ✓

Bounded: NO!

Hence, D is not compact $\Rightarrow$ Can't use EVT.

$$\mathcal{L} = x^2 + y^2 - \lambda(xy - 1)$$

3 eqns, 3 unknowns: x, y, λ

$$\boxed{\begin{aligned} \text{FOC: } \quad & \mathcal{L}'_x = 2x - \lambda y = 0 \\ & \mathcal{L}'_y = 2y - \lambda x = 0 \\ \text{C: } \quad & xy = 1 \end{aligned}}$$

$$2x = \lambda y$$

$$x = \frac{\lambda y}{2}$$

$$2y - \lambda\left(\frac{\lambda y}{2}\right) = 0 \quad \Big| \cdot 2$$

$$4y - \lambda^2 y = 0$$

$$y(4 - \lambda^2) = 0$$

3 cases:

$y = 0$:

$$x = \frac{\lambda \cdot 0}{2} = 0$$

C: $xy = 0 \neq 1$

Not a candidate pt. because the constraint doesn't hold

$\lambda = 2$:

$$x = \frac{2y}{2} = y$$

C: $xy = 1$

$x^2 = 1$

$x = \pm 1$

Candidadte points:

$(1, 1)$ $f = 2$ and

$(-1, -1)$ $f = 2$

$\lambda = -2$:

$$x = -y$$

C: $xy = 1$

$(-y)y = 1$

$\underbrace{y^2 = -1}$

NOT POSSIBLE!

$\downarrow$

No candidate points.

NOTE: Admissible pts with degenerate constraint?

$g(x,y) = xy$

$$\left.\begin{aligned} g'_x &= y = 0 \\ g'_y &= x = 0 \end{aligned}\right\} \Rightarrow xy = 0 \cdot 0 = 0 \neq 1$$

$\Rightarrow$ there are no admissible points with a degenerate constraint.

(holds in general for hyperbola-constraints)

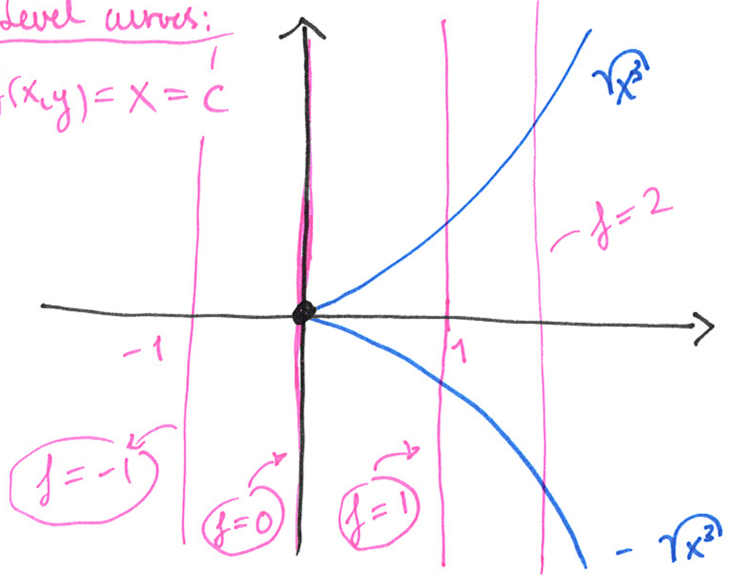
CONCLUSION: $f_{min} = 2$ at $(1,1)$ and $(-1,-1)$ with $\lambda = 2$.

No maximum since $y = \frac{1}{x}$ will satisfy the constrait. Can let $x \to \infty$. Then, $y \to 0$, but is admissible. Then, $f(x,y) = x^2 + y^2$

$$\to \infty^2 + 0^2 = \infty.$$

Ex: max / min $f(x,y) = x$ when $\underbrace{y^2 - x^3 = 0}_{D:\ y^2 - x^3 = 0}$

Level curves: $f(x,y) = x = c$



$$y = \pm \underbrace{\sqrt{x^3}}$$

NB: Only defined for $x^3 \geq 0$ $\Rightarrow x \geq 0$

Can we have admissible points with a degenerate constraint?

$$g'_x = -3x^2 = 0 \Rightarrow x = 0$$

$$g'_y = 2y = 0 \Rightarrow y = 0, \text{ but then}$$

$g(x,y) = g(0,0) = 0^2 - 0^3 = 0$, so $(0,0)$ is on D. Hence, $(0,0)$ is an admissible point with degenerate constraint.

Do we have any type i/ordinary candidate points?

$$\mathcal{L}(x,y) = x - \lambda(y^2 - x^3)$$

FOC:

$$\mathcal{L}'_x = 1 + \lambda \cdot 3x^2 = 0 \quad : (1)$$

$$\mathcal{L}'_y = -\lambda \cdot 2y = 0 \quad : (2)$$

C:

$$y^2 - x^3 = 0 \quad : (3)$$

(2): $-\lambda \cdot 2y = 0$

OR

$\lambda = 0$:

(1): $1 + 0 \cdot 3x^2 = 0$

$1 = 0$

Never true! $\Rightarrow$ No candidate point.

$y = 0$:

(3): $0^2 - x^3 = 0 \Rightarrow x = 0$

(1): $1 + \lambda \cdot 3 \cdot 0 = 0$

$1 = 0$

NEVER TRUE! $\Rightarrow$ No candidate.

Hence, we have no ordinary candidate points.

All we have: Admissible pt. with degenerate constraint, i.e., $(0,0)$.

From the Figure: We see that this is the minimum point. No maximum (no candidates or from figure)

D intersects level curves with higher & higher value

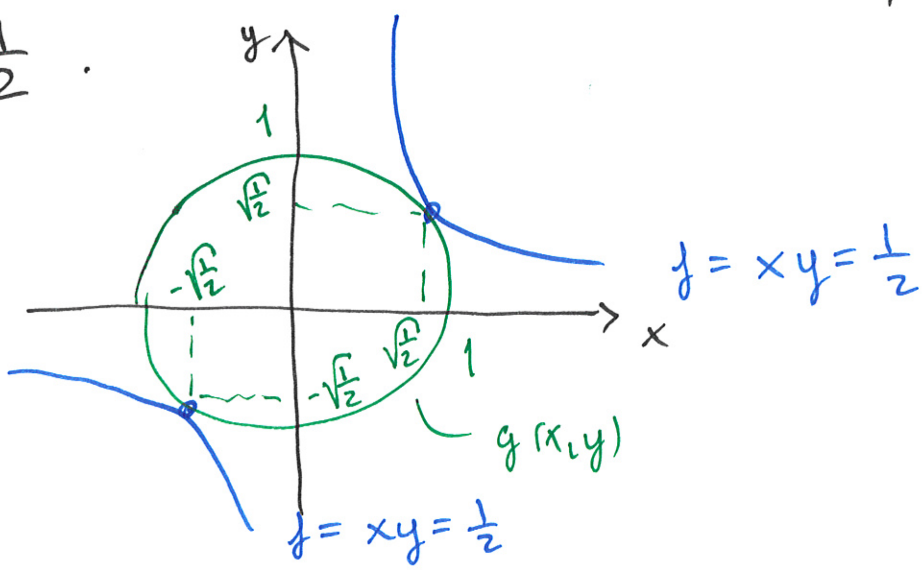
OR: $y = \pm\sqrt{x^3}$ makes constraint hold. Can choose arbitrarily large $x \Rightarrow f(x,y) = x \to \infty$ $\Rightarrow$ NO MAX

Interpretation of Lagrange multipliers

Ex: max/min $f(x,y) = xy$ when $x^2+y^2=1$

Shown: $f_{max} = \frac{1}{2}$ at $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

with $\lambda = \frac{1}{2}$.



NEXT WEEK: Think of radius of circle as parameter.