Lagrange problems

EBA 1180

Spring 23

- Lagrange problems = optimization problems (max / min) with equality constraints

$$(*) \quad \boxed{\max/\min\ f(x,y) \text{ when } \underbrace{g(x,y)}_{\text{function}} = \underbrace{a}_{\text{constant}}}$$

Ex: $\min f(x,y) = x^2 + y^2$ when $\underbrace{3x+4y}_{g(x,y)} = \underbrace{12}_{a}$

Draw: $4y = 12 - 3x \quad | : 4$

$y = 3 - \frac{3}{4}x$; $\quad$ $x=0$: $y = \frac{12}{4} = 3$

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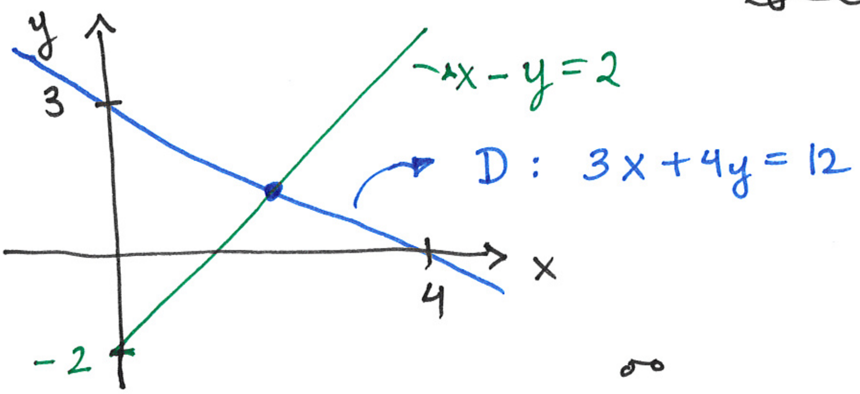


Ex: $\min f(x,y) = x^2 + y^2$ when $3x + 4y = 12$

and $x - y = 2$

$y = x - 2$

A point: Intersection of two lines (see Figure above)

Recall: General method:

1) Find candidate points

2) Determine whether any of these are max/min.

For Lagrange problems:

i) Interior stationary points; NONE

ii) Other interior critical points; NONE

iii) Boundary points

Extreme value theorem:

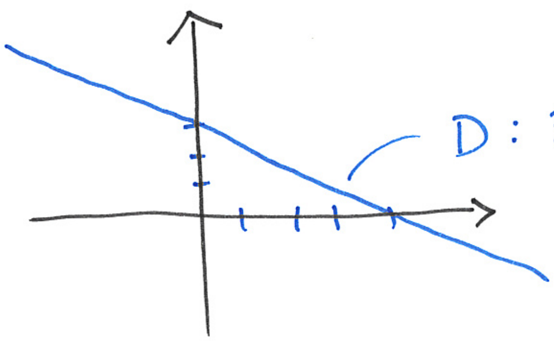
If D is compact (closed & bounded) and f is continuous, then f has a max/min on D

and

Always true for Lagrange problems (= constraint)

Not necessarily true for Lagrange problems

Ex: min $f(x,y) = x^2 + y^2$ when $3x + 4y = 12$



$D: 3x + 4y = 12$; D is closed, but not bounded $\Rightarrow$

D is NOT compact $\Rightarrow$ EVT cannot be used.

oo

Method of Lagrange multipliers

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\,(g(x,y) - a)$$

Lagrangian (Lagrange function)

Lagrange multiplier

$$= x^2 + y^2 - \lambda\,(3x + 4y - 12)$$

Example

Candidates for max/min: The stationary points of $\mathcal{L}$:

Example

FOC: $\mathcal{L}'_x = f'_x - \lambda g'_x = 0 = 2x - 3\lambda$

$\mathcal{L}'_y = f'_y - \lambda g'_y = 0 = 2y - 4\lambda$

C: $\mathcal{L}'_\lambda = -(g(x,y) - a) = 0 = -(3x + 4y - 12) = 0$

constraint

$g(x,y) - a = 0$ $\Leftrightarrow$ $\boxed{3x + 4y = 12}$

$g(x,y) = a$

The constraint

First order condition

Lagrange conditions: FOC + C

FOC: $\mathcal{L}'_x = 2x - 3\lambda = 0$: (1)

$\mathcal{L}'_y = 2y - 4\lambda = 0$: (2)

C: $3x + 4y = 12$: (3)

System of 3 eqns. & 3 unknowns: x, y, λ

From (1): $\quad 2x = 3\lambda \Rightarrow x = \frac{3}{2}\lambda \quad (\star)$

From (2): $\quad 2y = 4\lambda \Rightarrow y = 2\lambda \quad (\star\star)$

From C: $\quad 3\left(\frac{3}{2}\lambda\right) + 4(2\lambda) = 12 \quad \Big| \cdot 2$

$$9\lambda + 16\lambda = 24$$

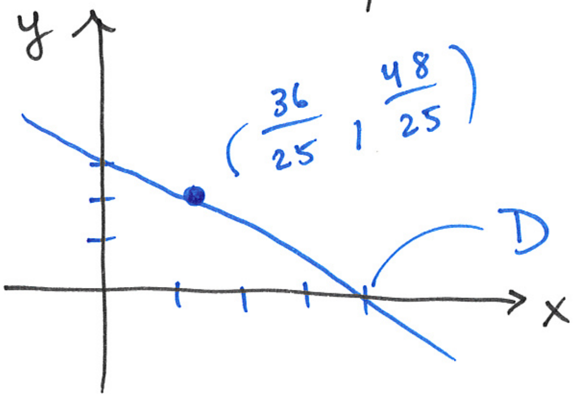
$$25\lambda = 24$$

$$\underline{\lambda = \frac{24}{25}}$$

$\Rightarrow (\star)$: $\quad x = \frac{3}{2} \cdot \frac{24}{25} = \frac{3 \cdot 12}{25} = \underline{\frac{36}{25}}$

$(\star\star)$: $\quad y = 2 \cdot \frac{24}{25} = \underline{\frac{48}{25}}$

Only one candidate point: $\left(\frac{36}{25}, \frac{48}{25}; \frac{24}{25}\right)$



Alternative method (substitution)

$$\boxed{\min f(x,y) = x^2 + y^2 \text{ when } 3x + 4y = 12}$$

$4y = 12 - 3x$

$(\star)$: $y = 3 - \frac{3}{4}x$

$$x^2 + y^2 = x^2 + \left(3 - \frac{3}{4}x\right)^2$$

$$= x^2 + 9 - \frac{9}{2}x + \frac{9}{16}x^2$$

$$= \frac{25}{16}x^2 - \frac{9}{2}x + 9 =: g(x)$$

Define a one-variable function $g(x)$

Alt: $\min g(x) = \frac{25}{16}x^2 - \frac{9}{2}x + 9$

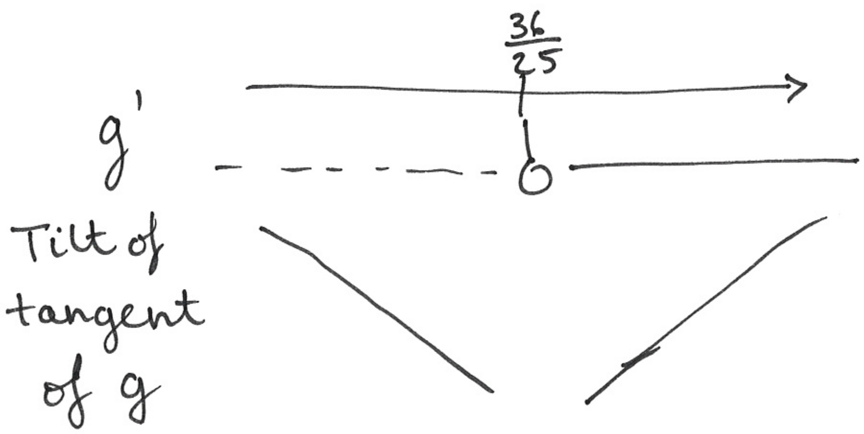
$$g'(x) = \frac{25}{16}\cdot 2x - \frac{9}{2} = 0$$

$$\frac{25}{8}x - \frac{9}{2} = 0 \quad \Big| \cdot 8$$

$$25x - 36 = 0$$

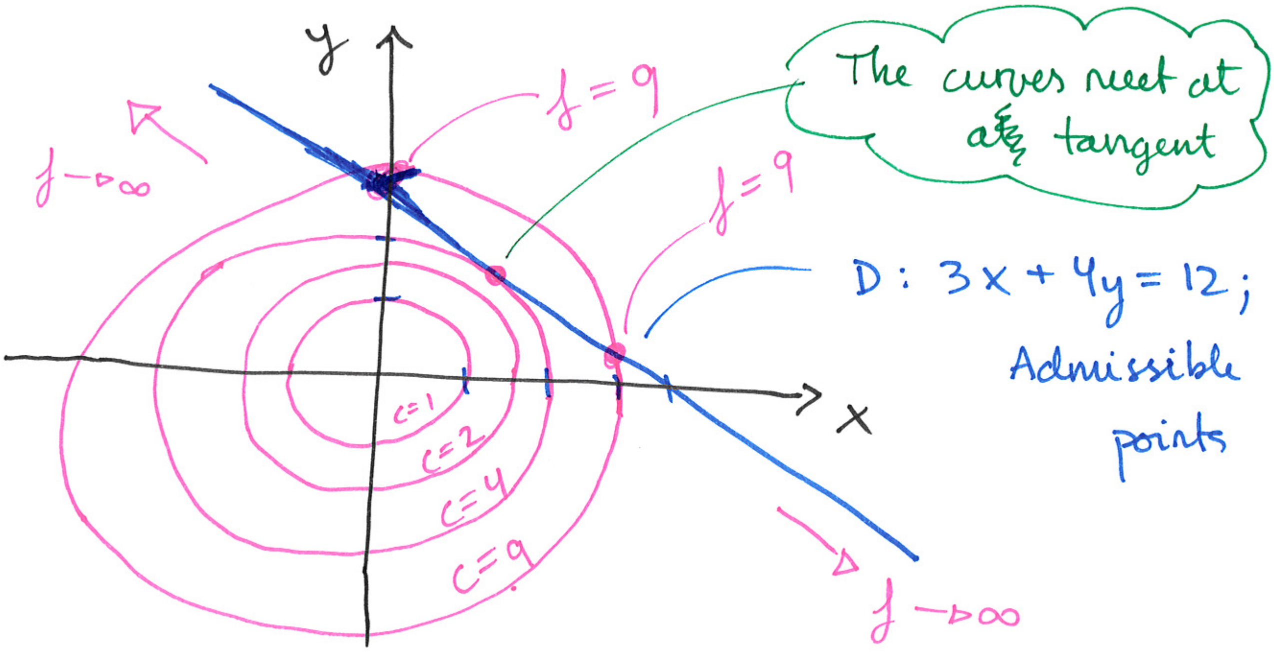
$$\underline{x = \frac{36}{25}}$$

(★): $y = 3 - \frac{3}{4}\cdot\frac{36}{25} = \dots = \underline{\frac{48}{25}}$



Hence, $x = \frac{36}{25}$ is a minimum for $\underline{\underline{g}}$.

Ex: max/min $f(x,y) = x^2+y^2$ when $3x+4y=12$



Level curves of f: $f(x,y) = c$

$x^2+y^2 = c$ → Circle, center $(0,0)$, $r = \sqrt{c}$, $c > 0$. If $c = 0$: A point $(0,0)$. No points if $c < 0$

$c=1$: $x^2+y^2=1$

$c=2$: $x^2+y^2=2$

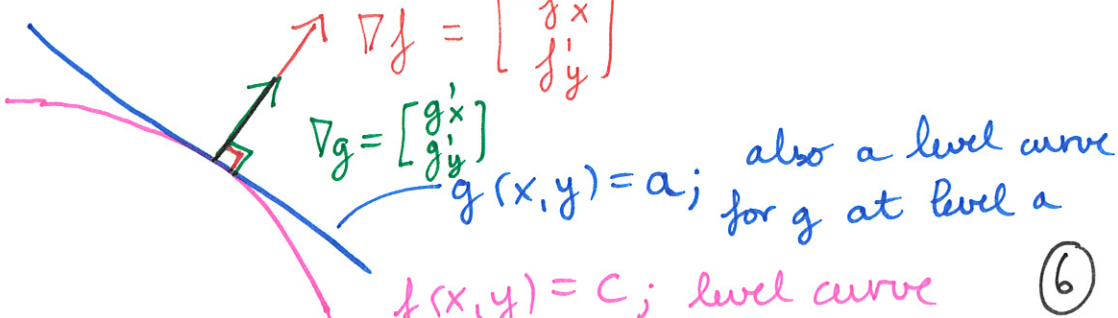
$c=4$: $x^2+y^2=4$

$c=9$: $x^2+y^2=9$

Candidates for max/min:

Points where the two curves meet at a tangent

$$\begin{cases} 3x+4y = 12 & : D \\ x^2+y^2 = c & : \text{Level curve} \end{cases}$$



Slopes of the tangents of level curves should be equal:

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

$$-\frac{2x}{2y} = -\frac{3}{4}$$

$$4x = 3y$$

$$y = \frac{4}{3}x$$

Constraint:

$$3x + 4\left(\frac{4}{3}x\right) = 12 \quad \Big| \cdot 3$$

$$9x + 16x = 36$$

$$25x = 36$$

$$\underline{x = \frac{36}{25}}$$

NOTE: $\nabla f = \lambda \nabla g$ (gradient of f is a scalar multiple of the gradient of g)

$$\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \lambda \begin{bmatrix} g'_x \\ g'_y \end{bmatrix}$$

FOC:

$$\begin{cases} f'_x = \lambda g'_x \\ f'_y = \lambda g'_y \end{cases} \Rightarrow \boxed{\begin{aligned} \mathcal{L}'_x &= f'_x - \lambda g'_x = 0 \\ \mathcal{L}'_y &= f'_y - \lambda g'_y = 0 \end{aligned}}$$

Theorem: If (x^*, y^*) is max/min in a Lagrange problem:

$$\boxed{\max/\min f(x,y) \text{ with } g(x,y) = a}$$

Then either

i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfies the Lagrange constraints FOC + C:

FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

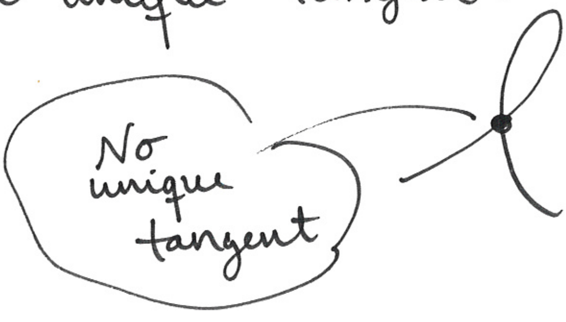
OR

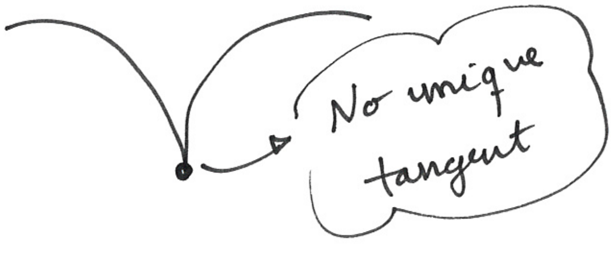
ii) The constraint is degenerate at (x^*, y^*), i.e.;

$g'_x = 0$ and $g(x,y) = a$

and $g'_y = 0$

Ex: In general, an extreme point with a degenerate constraint is a point where D does not have a unique tangent:





Ex: min $x^2 + y^2$ with $\underbrace{3x + 4y}_{g(x,y)} = 12$

$g'_x = 3 \neq 0$

$g'_y = 4 \neq 0$, so case ii) of the Theorem is not possible.