Examples: Optimization

EBA 1180
Spring 23

Warm-up: $f(x,y) = x^2 y^3 + y^2 - 2y$, $D_f = \mathbb{R}^2$

No boundary

An unconstrained optimization problem.

Q: max / min $f(x,y)$?

Candidate points:

- Stationary point(s): $f'_x = 0$
  $f'_y = 0$

- Other candidate points?

- Stationary points: $f'_x = 2xy^3 = 0 \Rightarrow x = 0$ or $y = 0$

$$f'_y = x^2 3y^2 + 2y - 2 = 0$$

2 cases:

$x=0$:	$y=0$:
$0 + 2y - 2 = 0$	$0 + 0 - 2 = 0$
$2y = 2$	$-2 = 0$
$y = 1$	Never true!

Hence, only one stationary point: $(x^*, y^*) = (0, 1)$

Note: $f(0,1) = \ldots = \underline{-1}$

- Other candidate points? Partial derivatives are defined everywhere and no boundary $\Rightarrow$ The stationary pt. is the only candidate point.

Classification of stationary point:

$$H(f)(x,y) = \begin{array}{ccc} f''_{xx} & & f''_{xy} \\ & \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{bmatrix} & \\ f''_{yx} & & f''_{yy} \end{array}$$

Insert $(0,1)$:

$$H(f)(0,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ so}$$

$$\det H(f)(0,1) = 2\cdot 2 - 0\cdot 0 = 4 > 0$$

$$\operatorname{tr} H(f)(0,1) = 2+2 = 4 > 0$$

$\Downarrow$ ~ Second derivative test

$f(0,1) = -1$ is a local minimum.

NOTE: If $x=1$, then:

$$f(1,y) = \underbrace{y^3+y^2-2y}$$

Can be made arbitrarily small (test on calc. if unsure: $y = -100, -1000, -10\,000$) OR be made arbitrarily large (try: $y = 100, 1000, 10\,000$).

$\Downarrow$

f has no (global) maximum or (global) minimum.

NB: Example of a function with only one stationary point, which is a local min., but still has no global minimum.

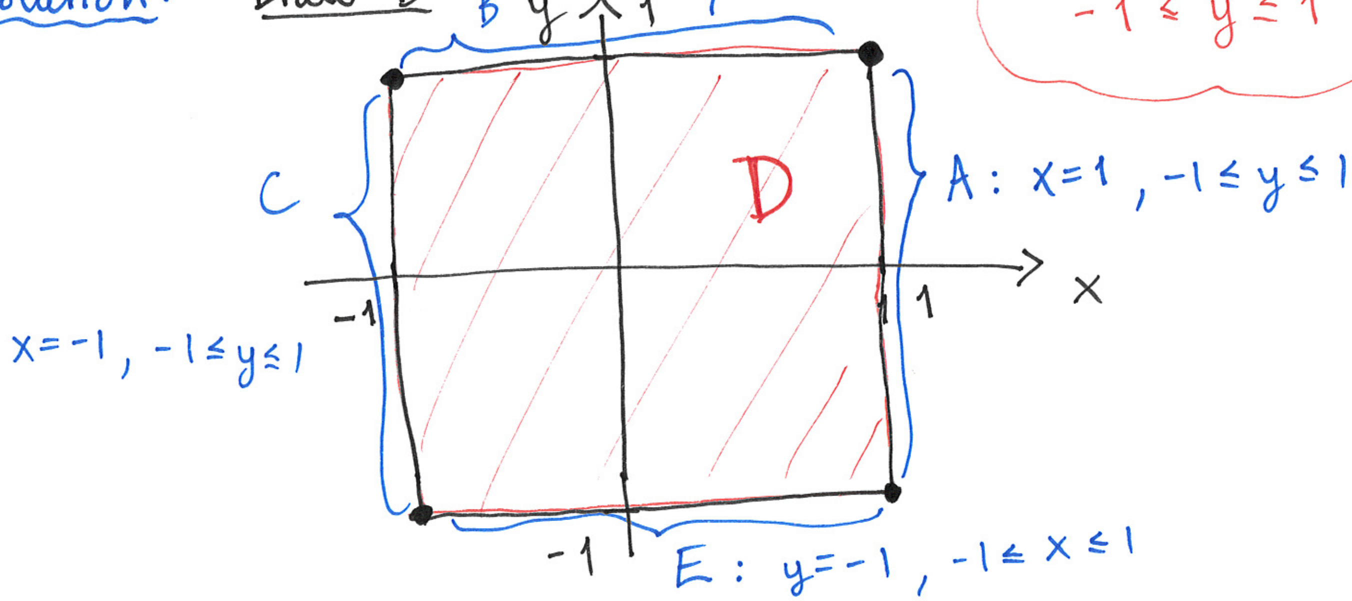
Exercise sheet 43

5) d) max/min $f(x,y) = xy(x^2 - y^2) = x^3y - xy^3$

when $\underbrace{-1 \le x, y \le 1}_{D}$.

$-1 \le x \le 1$ and $-1 \le y \le 1$

Solution: Draw D:



NOTE: f is continuous, D is closed ($\le$) and bounded (can be boxed in) $\Rightarrow$ D is compact. $\Rightarrow$ The extreme value theorem gives f has a max. and min. (over D).

Candidate points

i) Interior stationary points:

$$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \Rightarrow \begin{cases} f'_x = 3x^2 y - y^3 = 0 \\ f'_y = x^3 - 3xy^2 = 0 \end{cases}$$

(1): $y(3x^2 - y^2) = 0$

(2): $x(x^2 - 3y^2) = 0$

$3x^2 = y^2$

From (1): $y = 0$: or $3x^2 - y^2 = 0$:

(2): $x(x^2 - 3 \cdot 0^2) = 0$

$x^3 = 0$

$x = 0$

(2): $x(x^2 - 3 \cdot 3x^2) = 0$

$x(x^2 - 9x^2) = 0$

$-8x^3 = 0$

$x = 0$

$\Downarrow$

$y^2 = 3 \cdot 0^2 = 0$

$y = 0$

Only one interior stationary point: $(x, y) = (0, 0) \Rightarrow$

$$f(0,0) = 0$$

ii) Other interior critical points: No other such points since f'_x and f'_y are defined everywhere.

iii) Boundary of D:

A: $x=1$, $-1 \le y \le 1$:

$$f(x,y) = f(1,y) = y - y^3 =: h(y)$$

$$h'(y) = 1 - 3y^2 = 0$$

$$3y^2 = 1$$

$$y^2 = \frac{1}{3}$$

$$y = \pm \frac{1}{\sqrt{3}} \quad (\text{Between } -1 \text{ and } 1)$$

Candidates from A: $\left(1, -\frac{1}{\sqrt{3}}\right)$, $\left(1, \frac{1}{\sqrt{3}}\right)$, $(1,1)$, $(1,-1)$

$$f\left(1, -\tfrac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = -\frac{2}{3\sqrt{3}}$$

$$f(1,-1) = 0$$

$$f(1,1) = 0$$

boundary pts. of A viewed as a one-variable function: y-axis

$$f\left(1, \tfrac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}}$$

B: $y=1$, $-1 \le x \le 1$:

$$f(x,1) = x(x^2-1) =: h(x)$$
$$= x^3 - x$$

$$h'(x) = 3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \pm \frac{1}{\sqrt{3}}$$

Candidates from B: $\left(\frac{-1}{\sqrt{3}}, 1\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$,

$(1,1)$, $(-1,1)$

$f(1,1) = \underline{0}$

Boundary pts of B

$f(-1,1) = \underline{0}$

$f\left(\frac{-1}{\sqrt{3}}, 1\right) = -\frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \underline{\frac{2}{3\sqrt{3}}}$

$f\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \underline{-\frac{2}{3\sqrt{3}}}$

C : $x = -1,\ -1 \le y \le 1$:

$$f(-1, y) = -y(1 - y^2) = y^3 - y =: h(y)$$

$$h'(y) = 3y^2 - 1 = 0$$

$$y = \pm\frac{1}{\sqrt{3}}$$

$f\left(-1, -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \underline{\frac{2}{3\sqrt{3}}}$

Candidates from C: $\left(-1, -\frac{1}{\sqrt{3}}\right)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$,

$(-1, 1)$, $(-1,-1)$.

$f(-1,1) = \underline{0}$

Boundary pts. of C

$f(-1,-1) = \underline{0}$

$f\left(-1, \frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\right) = \underline{-\frac{2}{3\sqrt{3}}}$

E: $y=-1,\ -1 \le x \le 1$:

$$f(x,-1) = -x\,(x^2-1) = x - x^3 =: h(x)$$

$$h'(x) = 1 - 3x^2 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

$f\left(-\frac{1}{\sqrt{3}},-1\right) = -\frac{2}{3\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}},-1\right) = \frac{2}{3\sqrt{3}}$

Candidates from E: $\left(-\frac{1}{\sqrt{3}},-1\right)$, $\left(\frac{1}{\sqrt{3}},-1\right)$, $(-1,-1)$, $(1,-1)$

$f(-1,-1)=0$

$f(1,-1)=0$

To conclude: Compare function values of all of the candidates. Know min/max (over D) exists from EVT. Hence,

Maximum:

$$f_{max} = \frac{2}{3\sqrt{3}}$$

in the maximum points ~~$\left(1,-\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}},1\right)$~~, ~~$\left(-1,\frac{1}{\sqrt{3}}\right)$~~ $\left(1,\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}},1\right)$, $\left(-1,-\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}},-1\right)$.

Minimum:

$$f_{min} = -\frac{2}{3\sqrt{3}}$$

in the minimum points $\left(1, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$,

$\left(-1, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -1\right)$.

NB: $$\frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{3\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3 \cdot 3} = \frac{2\sqrt{3}}{9}$$

$$-\frac{2}{3\sqrt{3}} = -\frac{2\sqrt{3}}{9}$$