EBA 1180
Spring 23

# Recap: Dot product / inner product

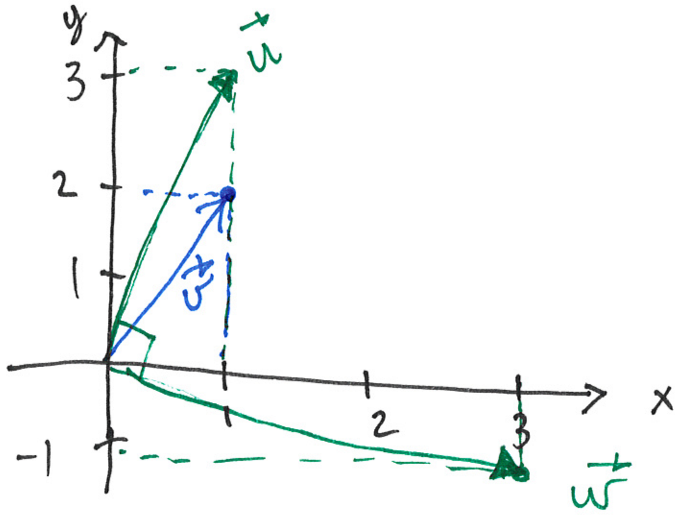
$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

**Ex:** $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{\underline{1}}$$

$$\vec{u} \cdot \vec{w} = 3 \cdot 1 + (-1) \cdot 3 = 3 - 3 = \underline{\underline{0}}$$




NOTE: $\vec{u} \cdot \vec{w} = 0$

and $\vec{u} \perp \vec{w}$

has a $90°$ angle with

"$\vec{u}$ and $\vec{w}$ are orthogonal"

**Result:** $\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$

## Rules of computation:

1) $\vec{v} \cdot \vec{w} = $ a number

2) $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \ldots + v_n^2 = \|\vec{v}\|^2$

Recall def. $\|\vec{v}\|$

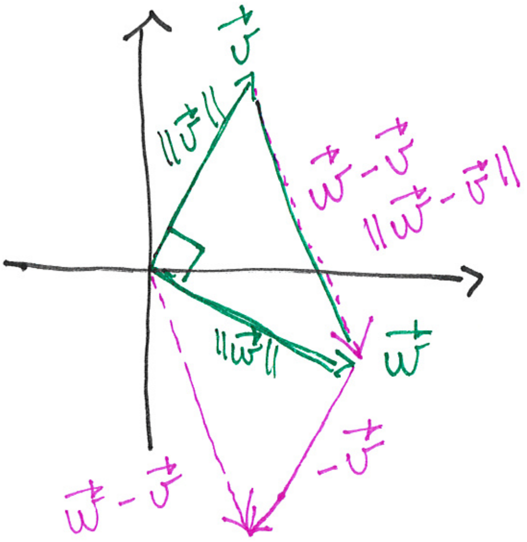
3) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

$(a\vec{u} + b\vec{v}) \cdot \vec{w} = a\vec{u} \cdot \vec{w} + b\vec{v} \cdot \vec{w}$

## Proof of Result:

Pythagoras

$$\vec{v} \perp \vec{w} \iff \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{w} - \vec{v}\|^2$$




$$\Updownarrow$$

$$(\cancel{v_1^2} + \ldots + \cancel{v_n^2}) + (\cancel{w_1^2} + \ldots + \cancel{w_n^2})$$

$$= \underbrace{(w_1 - v_1)^2} + \ldots + \underbrace{(w_n - v_n)^2}$$

$$= \underbrace{\cancel{w_1^2} - 2w_1 v_1 + \cancel{v_1^2}} + \ldots +$$

$$\underbrace{\cancel{w_n^2} - 2w_n v_n + \cancel{v_n^2}}$$

$$0 = -2w_1 v_1 - \ldots - 2w_n v_n \quad \Big| :(-2)$$

$$\Updownarrow$$

$$w_1 v_1 + \ldots + w_n v_n = 0$$

$$\Updownarrow$$

$$\vec{w} \cdot \vec{v} = 0$$

Def. of inner product

$$\Updownarrow$$

$$\vec{v} \cdot \vec{w} = 0$$

$\boxtimes$

NOTE: $\underbrace{\vec{v} \cdot \vec{w}}_{\substack{\text{inner prod. of} \\ n\text{-vectors}}} = \underbrace{\vec{v}^T \vec{w}}_{\text{matrix multiplication}}$

Ex: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 2 \cdot 1 + 1 \cdot (-3) = \underline{-1}$$

$$\parallel$$

$$\vec{v}^T \vec{w} = \underset{1 \times \cancel{2}}{\begin{bmatrix} 2 & 1 \end{bmatrix}} \cdot \underset{\cancel{2} \times 1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}} = 2 \cdot 1 + 1(-3) = \underline{-1}$$

NB: $\underbrace{\vec{v}\,\vec{w}}_{\substack{\text{matrix} \\ \text{multiplication}}}$ is not defined: $\underset{2 \times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \cdot \underset{2 \times 1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}$

$1 \neq 2 \Rightarrow$ not def.

Application of lin. alg. + stochastic processes: Google page rank algorithm!

## Functions in two variables

Ex: $f(x, y) = 2x + 3y - 1$, linear function

$f(x, y) = x^2 + y^2$, polynomial function

$f(x, y) = \dfrac{x+y}{x-y}$, rational function

$f(x, y) = x e^y$

- General:

$f(x,y)$ : functional expression in $x, y$

$x, y$ : input variables

$z = f(x,y)$ : output variable

**Def (Domain of $f$):**

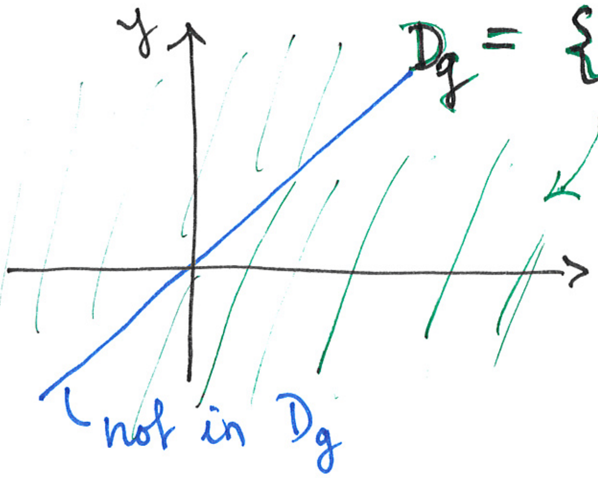
$D_f$ = domain of $f$ = all coordinate pairs $(x,y)$ that we can use as inputs to the function $f$

Subset of the $xy$-plane; $\mathbb{R}^2$

Ex: $f(x,y) = 2x + 3y - 1$, $D_f = \mathbb{R}^2$

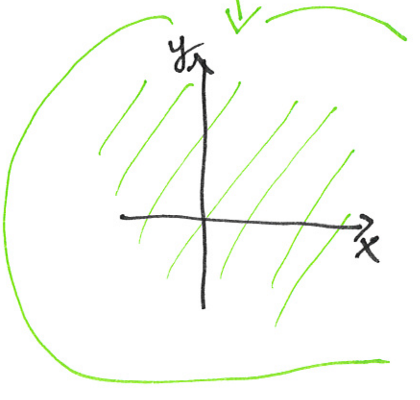
$g(x,y) = \dfrac{x+y}{x-y}$, $D_g$ : $x - y \neq 0$

$D_g = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$




$y \neq x$

$y = x$



**Def (Range):**

$V_f$ = range of $f$ = all values $\cancel{\ }$ $f(x,y)$ can attain when $(x,y) \in D_f$

$(x,y)$ is in $D_f$

- To find the range, $V_f$: Find the max/min. of $f$.

Both x, y can be arbitrarily large/small

Ex: $f(x,y) = 2x + 3y - 1$, $V_f = (-\infty, \infty) = \mathbb{R}$

$f(x,y) = x^2 + y^2$, $V_f = [0, \infty)$

Can only get non-neg. values because of squares

## Graphs and level curves

**Def: (Graph of function in two variables)**

The graph of a function $f$ in two variables is the set of all points

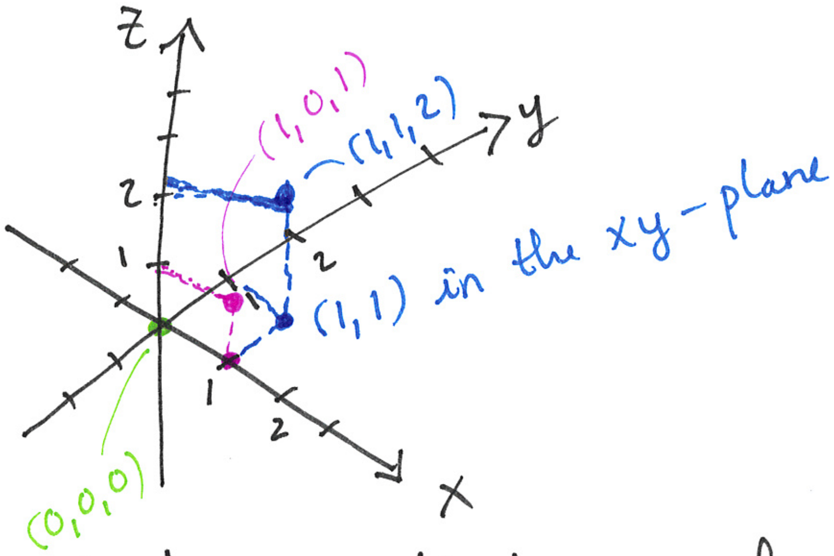
$$(x, y, z)$$

where $(x,y) \in D_f$ and $z = f(x,y)$.

- Can draw the graph of $f$ in the $xyz$-coordinate system.

Ex: $f(x,y) = x^2 + y^2$, $D_f = \mathbb{R}^2$

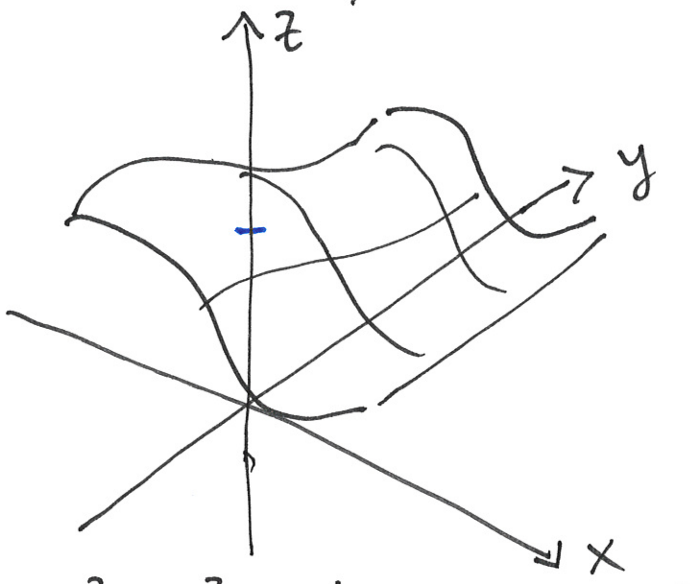
| $(x,y)$ | $(0,0)$ | $(1,0)$ | $(1,1)$ |
| --- | --- | --- | --- |
| $z = f(x,y)$ | 0 | 1 | 2 |
| Point in the $xyz$-plane | $(0,0,0)$ | $(1,0,1)$ | $(1,1,2)$ |




- The graph of $f$ is called a surface.

**Def (Level curves):** All $(x,y)$ such that $f(x,y) = c$ for a constant $c$.

- In general: Graph of ~~of~~ a function $f(x,y)$:




Ex: $f(x,y) = x^2+y^2$. Level curves?

$c=2$: $f(x,y)=2$

$x^2+y^2=2 \rightsquigarrow$ circle, center $(0,0)$, $r=\sqrt{2}$

$c=1$: $f(x,y)=1$

$x^2+y^2=1 \rightsquigarrow$ circle, center $(0,0)$, $r=1$

$c=0$: $f(x,y)=0$

$x^2+y^2=0 \rightsquigarrow \Leftrightarrow x=y=0 \Rightarrow$

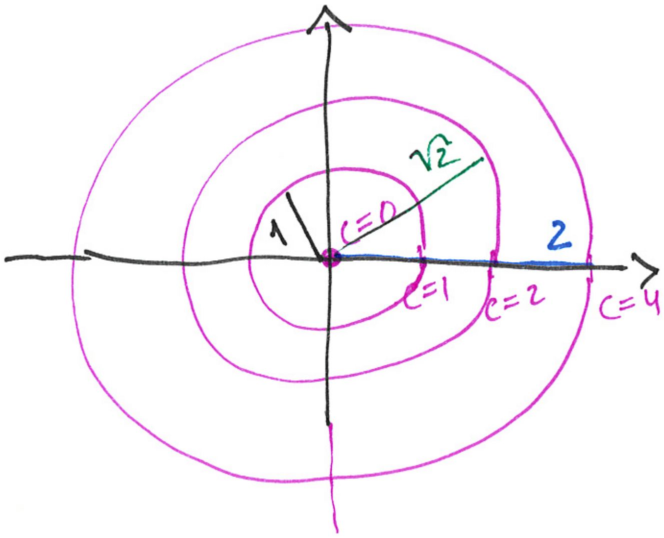
Level "curve" is just a point: $(0,0)$.

$c=4$: $x^2+y^2=4 \rightsquigarrow$ circle, center $(0,0)$, $r=2$

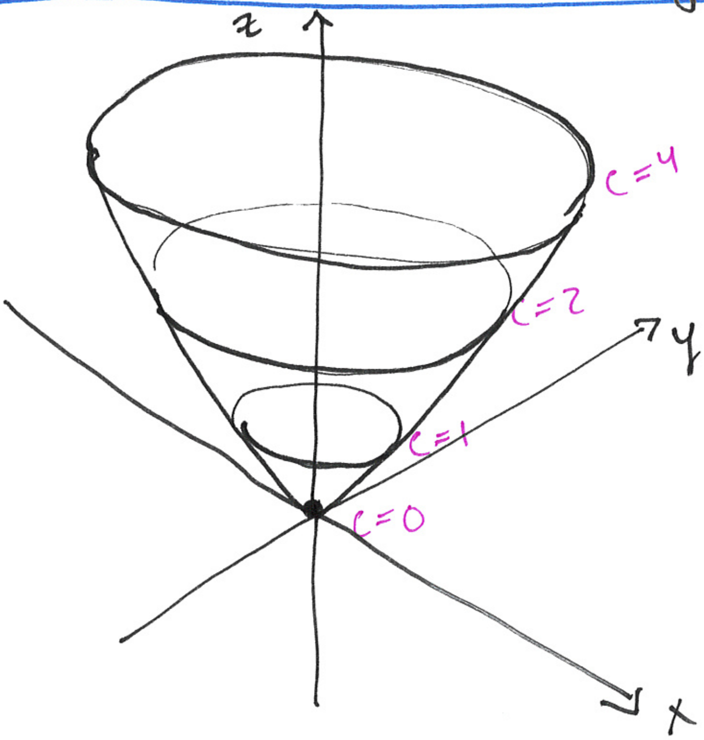
$c=-1$: $x^2+y^2=-1 \rightsquigarrow$ no such points

Illustration of the level curves from above:

Shown in xy-plane




→ Use level curves to draw the graph of $f$:




$$f(x,y) = x^2 + y^2$$