Determinants and linear systems

EBA1180 Spring 23

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} \ddots \end{vmatrix}$$

Cofactor expansion:

$$+ 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} - 0 \begin{vmatrix} \ddots \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \cancel{- 0 \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}} + 1 \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

Better choice: row 2

$$\cancel{+ 0 \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}} - 1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1(1-0) + 1(0+1) - 1(-1-0) + 1(1-0)$$

$$= 1 + 1 + 1 + 1 = \underline{\underline{4}}$$

Alternative method for finding the determinant

1) Gaussian elimination until upper triangular matrix. → NB: May be scaling and/or change of signs
2) Determinant is then the product of the diagonal elements

Ex: $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \underset{\substack{(-1)*\text{row }1 \\ \text{and } \cancel{\text{to}} \text{ row } 3}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$\underset{\substack{(-1)*\text{row }2 \\ \text{add to row } 4}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = E$

RESULT: If E is an upper triangular matrix (i.e., all entries below main diagonal are 0), then $|E|$ is the product of the diagonal entries.

NOTE: All echelon forms are upper triangular.

Ex ctd: i) $|E| = 1 \cdot 1 \cdot (-2) \cdot (-2) = \underline{\underline{4}}$

ii) NOTE: Here $|A| = |E| = 4$

Why does the result hold?

Ex:

$$|E| = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - 0 \begin{vmatrix} \ddots \end{vmatrix}$$

Cofactor expansion

$$+ 0 \begin{vmatrix} \ddots \end{vmatrix} - 0 \begin{vmatrix} \ddots \end{vmatrix}$$

$$= 1 \cdot 1 \cdot \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot \left((-2) \cdot (-2) - 0 \right)$$

$$= 1 \cdot 1 \cdot (-2) \cdot (-2) = 4$$

Pattern holds for any upper triangular matrix.

Ex:

$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot 2 \cdot 0 = \underline{0}$$

RESULT: (Change in determinant from elementary row operations)

If $A \sim B$ via elementary row operations, then

i) Switch two rows $\Rightarrow |B| = -|A|$

ii) Multiply a row by $c \neq 0 \Rightarrow |B| = c|A|$

iii) Add a multiple of a row to another row:

$$|B| = |A|$$

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $|A| = 0 \cdot 1 - 1 \cdot 1 = \underline{-1}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$$

switch row 1 and row 2

10.57

$$|B| = 1 \cdot 1 - 1 \cdot 0 = \underline{1}$$
$$= -|A|$$

Result: i) $|A| \neq 0 \Rightarrow$ One (unique) solution.

ii) $|A| = 0 \Rightarrow$ no solutions <u>OR</u> ∞ many solutions.

Ex:
$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}\right\} (\star)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2$$

$= 2 \neq 0$, so $|A| \neq 0 \Rightarrow (*)$ has one unique solution (from Result)

Check via row reduction:

pivot

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix}$$

$(-1)\cdot$row 1 add to row 2
$(-1)\cdot$ row 1 add to row 3

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$(-2)\cdot$row 2 add to row 3

pivots

Pivots in each of diag. elements: So unique solution.

Ex. alt:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ 2x + 3y + 5z &= 10 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$= 10 - 12 - (5-8) + 3 - 4$

$= -2 + 3 - 1 = \underline{0} \Rightarrow$ no solutions OR infinitely many solutions

Check via row reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 2 & 3 & 5 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array}\right]$$

(-1) · row 1 add to row 2
(-2) · row 1 add to row 3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \infty \text{ many solutions}$$

(-1) · row 2 add to row 3

Linear systems with parametres

Ex:

$x + y = 4$

$x + ay = 6$

Here:

x, y: variables — endogeneous

a: parameter $\leadsto$ solution depends on this parameter

exogeneous

Solve :

Alt. 1 : Gaussian elimination

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & a & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

(-1) · row 1
add to row 2

Pivot or not ?

Depends !

Two cases:

$a = 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right]$$

$\Downarrow$

No solutions !

$a \neq 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & a-1 & 2 \end{array}\right]$$

Pivot!

$$x + y = 4$$

$$(a-1)y = 2$$

$$y = \frac{2}{a-1}$$

$$x = 4 - y = 4 - \frac{2}{a-1}$$

$$= \frac{4a-4-2}{a-1}$$

$$= \frac{4a-6}{a-1}$$

solution: $(x, y) = \left(\frac{4a-6}{a-1}, \frac{2}{a-1}\right), \ a \neq 1$

Ex: $x + y = 4$
$x + ay = 6$

Let's solve via Cramer's rule!

$$A = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}, \quad 2 \times 2 \text{ (square)}$$

$$\vec{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad |A| = a - 1$$

2 cases:

1) $a = 1$: $|A| = 0$

$\Rightarrow$ No solutions or infinitely many

2) $a \neq 1$: $|A| \neq 0 \Rightarrow$

One solution.

Use Cramer's rule to find it!

$$|A| = a - 1, \quad |A_1(\vec{b})| = \begin{vmatrix} 4 & 1 \\ 6 & a \end{vmatrix} = \underline{4a - 6}$$

$$|A_2(\vec{b})| = \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} = 6 - 4 = \underline{2}$$

From Cramer's rule:

$$x = \frac{|A_1(\vec{b})|}{|A|} = \underline{\underline{\frac{4a-6}{a-1}}}$$

$$y = \frac{|A_2(\vec{b})|}{|A|} = \underline{\underline{\frac{2}{a-1}}}$$