How to determine the number of solutions of a lin. system?

EBA 1180
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Ex:

$$x + y + z = 4$$
$$x - y + z = 2$$
$$x + 5y + z = 8$$

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 1 & 5 & 1 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 4 & 0 & 4 \end{array}\right]$$

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$

$(-1)\cdot$ row 1 add to row 2 and row 3

PIVOTS: Basic variables: x, y

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2\cdot$ row 2 add to row 3

Column without pivot: Free variable: z

Echelon form!

NO PIVOT! $\Rightarrow$ Free variable!

$$x + y + z = 4 \longrightarrow x + 1 + z = 4 \Rightarrow x = 3 - z$$
$$-2y = -2 \Rightarrow y = 1$$

Solution: $(x, y, z) = (3 - z, 1, z)$ where z is free (can be any number & the lin. syst. still holds as long as x, y as above.

So this linear system has <u>infinitely many solutions</u>.

<u>One more case:</u>

$$\begin{aligned} x+y+z&=4 \\ x-y+z&=2 \\ x+5y+z&=9 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 1 & 5 & 1 & 9 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} x+y+z&=4 \\ -2y\quad&=-2 \\ 0\cdot x+0\cdot y+0\cdot z&=1 \rightsquigarrow 0=1 \end{aligned}$$

NEVER TRUE!

$\Downarrow$

$\boxed{\text{NO SOLUTIONS!}}$

<u>In general:</u> Pivot in the last column of an (extended) echelon form $\Leftrightarrow$ no solutions.

<u>Def (Pivot position):</u> A <u>pivot position</u> is a position where there is a pivot in the echelon form.

Ex:
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ x_1 - 2x_2 - x_3 + 4x_5 &= 8 \\ 2x_1 + x_2 - 5x_3 + 7x_4 &= 11 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 1 & -2 & -1 & 0 & 4 & 8 \\ 2 & 1 & -5 & 7 & 0 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -3 & -2 & -1 & 3 & -9 \\ 0 & -1 & -7 & 5 & -2 & -23 \end{array}\right]$$

Add $(-1)\cdot$ row1 to row 2
—"— $(-2)\cdot$ row 1 to row 3

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & -3 & -2 & -1 & 3 & -9 \end{array}\right]$$

switch rows 2 & 3

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & 0 & 19 & -16 & 9 & 60 \end{array}\right]$$

$(-3)\cdot$ row 2 add to row 3

Echelon form!

Pivots! Pivot positions are:

$(1,1)$, $(2,2)$, $(3,3)$.

The lin. syst. has <u>two degrees of freedom</u> (x_4, x_5 free)

Hence, infinitely many solutions.

Why?

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ -x_2 - 7x_3 + 5x_4 - 2x_5 &= -23 \\ 19x_3 - 16x_4 + 9x_5 &= 60 \end{aligned}$$

$$\Downarrow$$

$$19x_3 = 60 + 16x_4 - 9x_5$$

$\rightsquigarrow$ Can choose any x_4, x_5 and the original lin. syst. still holds.

RESULT: For any linear system the pivot positions determine the number of solutions.

Different cases:

i) Pivot position in the last column: No solutions.

Ex: $\left[\begin{array}{ccc|c} \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right]$ or smth.

$1 \rightarrow$ non-zero

ii) No pivot position in the last column: The lin. syst. has solutions. How many?

a) Pivot pos. in all variable columns: One solution.

Ex: $\left[\begin{array}{ccc|c} 1 & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \\ 0 & 0 & 1 & \cdot \end{array}\right]$

Ex: $\left[\begin{array}{cccc|c} 1 & - & - & - & \vdots \\ 0 & 1 & - & - & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \end{array}\right]$

b) There are variable columns without pivot positions: Infinitely many solutions.

Theorem: Any linear system has either

i) No solutions. ⇝ Inconsistent

ii) One unique solution.

iii) Infinitely many solutions.

(ii) and (iii) ⇝ Consistent

Computations with matrices and vectors

Def: ($m \times n$ matrix)

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Ex:

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{bmatrix}}_{3 \text{ columns}} \Big\} \text{ 2 rows}$$

NB: capital letters to denote matrices

2×3 matrix

a_{13} ← column 3

↑ row 1

some number

$$A = \begin{array}{c c} & \begin{array}{ccc} 1: & 2: & 3: \end{array} \\ \begin{array}{c} 1: \\ 2: \end{array} & \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{3 \text{ columns}} \end{array} \Big\} \text{ 2 rows}$$

2×3 matrix

- Addition : $A + B$
- Subtraction : $A - B$

$\}$ Result: matrix of same size

Defined if A and B are the same size (e.g. both $m \times n$ / 2×3)

- Scalar multiplication: $r \cdot A$ $\}$ result: Matrix same size as A

Always defined

r : scalar (i.e., number)

A : matrix

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+(-1) & 3+1 \\ -1+1 & 0+2 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

Do addition/subtraction position by position.

Ex:

$$2 \cdot \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 4 \\ 2\cdot(-1) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -2 & 4 \\ 0 & 2 \end{bmatrix}$$

Do $\boxed{\text{scalar}}$ multiplication by scalar position by position.

Def (n-vector):

So: Vector is a particular kind of matrix

An n-vector is a matrix with n rows and 1 column (a column vector).

- Write vectors as: $\vec{v}$ = boldface v / $\mathbf{v}$ = $\underline{v}$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; 3-vector viewed as a column vector

($\vec{w} = [1 \quad 2 \quad 3]$; 3-vector viewed as a row vector)

Vector operations:

→ Addition: $\vec{v} + \vec{w}$
→ Subtraction: $\vec{v} - \vec{w}$
} Defined for vectors of same size

→ Scalar multiplication: $r \cdot \vec{v}$ (r scalar/number)
always def.

Ex:
ADD: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}}$

SUBTRACT: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2-(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}}$

SCALAR MULTIPLICATION:

$$2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}}$$

$$(-1) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1) \cdot 1 \\ (-1) \cdot 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}}$$