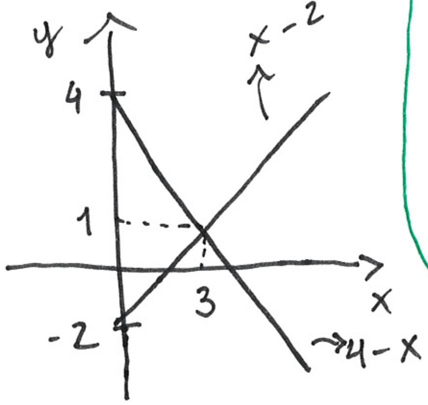
Systems of equations

EBA 1180
Spring23

Some kinds of systems of equations

LINEAR:

i) $x + y = 4$
$x - y = 2$



$x + y = 4 \Rightarrow y = 4 - x$

$x - y = 2 \Rightarrow y = x - 2$

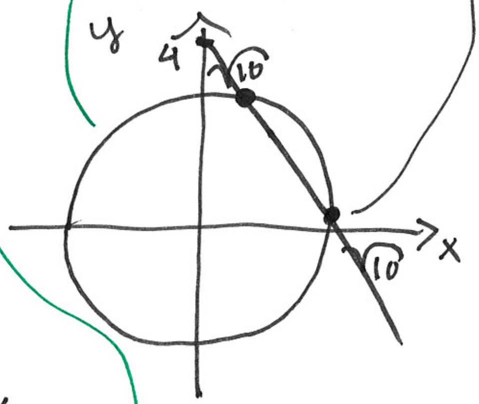
i): 2 methods

Eliminate	Substitute
$x + \cancel{y}$	$x+y=4 \Rightarrow$
$+ x - \cancel{y} = 4+2$	$y = 4 - x$
$2x = 6$	$x - y = 2$
$\underline{x = 3}$	$x - (4-x) = 2$
$y = 4 - 3 = \underline{1}$	$2x - 4 = 2$
$\underline{(x,y) = (3,1)}$	$2x = 6$
	$\underline{x = 3}$
	$y = 4 - 3 = 1$
	$\underline{(x,y) = (3,1)}$

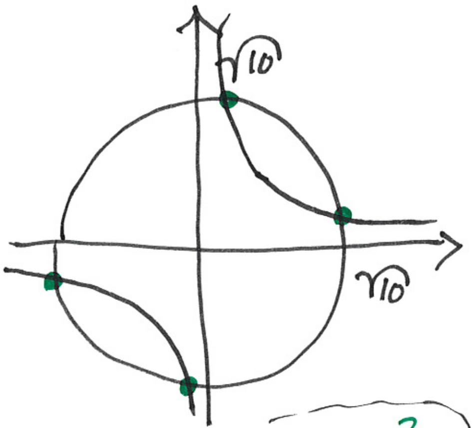
same!

NON-LINEAR:

ii) $x^2 + y^2 = 10$
$x + y = 4$



iii) $x^2 + y^2 = 10$
$xy = 3$



SOLVE:

ii)

$x + y = 4 \Rightarrow y = 4 - x$

$x^2 + (4-x)^2 = 10$

$x^2 + 16 - 8x + x^2 = 10$

$2x^2 - 8x + 16 = 10$

$x^2 - 4x + 8 = 5$

$x^2 - 4x + 3 = 0$

$x = \dfrac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 3}}{2}$

$= \dfrac{4 \pm 2}{2}$

$\underline{x_1 = 3}, \quad \underline{x_2 = 1}$

$y_1 = 4 - 3 = \underline{1}$

$y_2 = 4 - 1 = \underline{3}$

$\Downarrow$

$(x_1, y_1) = \underline{(3,1)}, \ (x_2, y_2) = \underline{(1,3)}$

$xy = 3 \Rightarrow \boxed{y = \dfrac{3}{x}}$

iii) $x^2 + \left(\dfrac{3}{x}\right)^2 = 10$

$x^2 + \dfrac{9}{x^2} = 10$

$x^4 + 9 = 10x^2$

$x^4 - 10x^2 + 9 = 0$

$(\overbrace{x^2}^{u})^2 - 10(\overbrace{x^2}^{u}) + 9 = 0$

Quad. formula: $x_1^2 = 9, \ x_2^2 = 1$

$\Downarrow$

$\underline{x_1 = \pm 3}, \quad \underline{x_2 = \pm 1}$

$\{(x,y)\} = \{(3,1), (-3,-1), (1,3), (-1,-3)\}$

Def: (Linear system)

An $m \times n$ linear system in the ~~vars~~ variables $x_1, x_2, \ldots, x_n$ is a system of m linear eqns. in n variables. It has the form:

$$\left.\underbrace{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}}_{n \text{ variables}}\right\} m$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are given numbers.

Ex:

$$\text{3 eqns}\left\{\underbrace{\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 7 \\ x_1 + x_2 \qquad\quad + 2x_4 &= 10 \\ x_1 + x_2 - x_3 \qquad\quad &= 3 \end{aligned}}_{4 \text{ variables}}\right.$$

$\rightsquigarrow$ 3×4 linear system.

Ex:

$$\left.\begin{aligned} x+y+z &= 3 \quad : 1) \\ x+2y+4z &= 7 \quad : 2) \\ \underbrace{x+3y+9z}_{\text{3 variables}} &= 13 \quad : 3) \end{aligned}\right\} \text{3 equations}$$

$\rightsquigarrow$ 3×3 linear system

Methods for solving linear systems

Substitution

1) $x = 3 - y - z$

2) $3 - y - z + 2y + 4z = 7$

$$\boxed{y + 3z = 4} \Rightarrow y = 4 - 3z$$

3) $3 - y - z + 3y + 9z = 13$

$$\boxed{2y + 8z = 10}$$

$\hookrightarrow$ $2(4 - 3z) + 8z = 10$

$8 - 6z + 8z = 10$

$z = 1$

$y = 4 \cdot 3 \cdot 1 = 1$

From 1): $x = 3 - 1 - 1 = 1$

Solution: $(x, y, z) = (1, 1, 1)$

Gaussian elimination

General and systematic method for solving *any* linear system.

METHOD

to be explained

1) Write down the augmented matrix of the linear system.

t.b.e

2) Use elementary row operations until you have an echelon form.

t.b.e

3) Use back substitution to solve the system.

Ex:

$$\left.\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}\right\} \quad 3 \times 3 \text{ linear system}$$

Write coefficients in a bot of numbers ⇝ matrix

1)

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right]}_{\text{augmented matrix of the linear system}}$$

Elementary row operations:

i) Switch two rows.

ii) Multiply a row with a constant $c \neq 0$.

iii) Add a multiple (by a constant) of a row to another row.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \overset{\text{row equivalent to}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right]$$

(-1)*(row 1) add to (row 2)

$[-1 \; -1 \; -1 \mid -3]$

$$\sim \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

(-1)*(row 1) add to (row 3)

$y + 3z = 4$
$2y + 8z = 10$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

(-2)*(row 2) add to (row 3)

$[0 \; -2 \; -6 \mid -8]$

$y + 3z = 4$
$2z = 2$

Echelon form!

Def: (Pivot)

The first non-zero element in a row is called a pivot.

Def: (Echelon form): An echelon form is where

1) All entries below a pivot are zero.
2) If some rows are all zeros, they are at the bottom of the matrix.

∞

- To solve a system in echelon form:

Back substitution;

1) Start with last equation
2) Work backwards and substitute the variables we have solved for.

$$\begin{aligned} x + y + z &= 3 \Rightarrow x = 3 - y - z = 3 - 1 - 1 = \underline{1} \\ y + 3z &= 4 \Rightarrow y = 4 - 3z \Rightarrow y = 4 - 3 = \underline{1} \\ 2z &= 2 \Rightarrow \underline{z = 1} \end{aligned}$$

one solution: $\underline{(x, y, z) = (1, 1, 1)}$