EBA 1180
Lect. 30

Spring 23

Today: Applications of the definite integral

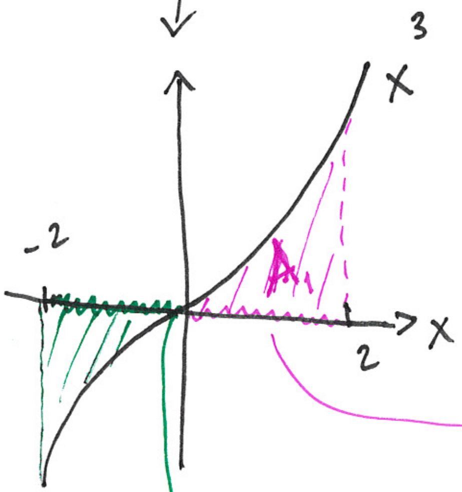
Calculating the area between graphs

What abt. int. of functions that are not always ≥ 0?

Ex: $\displaystyle\int_{-2}^{2} x^3 dx = \left[\frac{1}{4}x^4\right]_{x=-2}^{2}$

$= \frac{1}{4}\left(2^4 - (-2)^4\right)$

$= \frac{1}{4}(16 - 16) = 0$ why? (*)



NB: x^3 can be both pos. & neg. on $[-2, 2]$.

$I_1 = \displaystyle\int_0^2 x^3 dx = \left[\frac{1}{4}x^4\right]_{x=0}^{2}$

$= \frac{1}{4}(2^4 - 0^4) = 4 = A_1$

$I_2 = \displaystyle\int_{-2}^{0} x^3 dx = \left[\frac{1}{4}x^4\right]_{x=-2}^{0} = \frac{1}{4}\left(0^4 - (-2)^4\right)$

$= \frac{-16}{4} = -4$

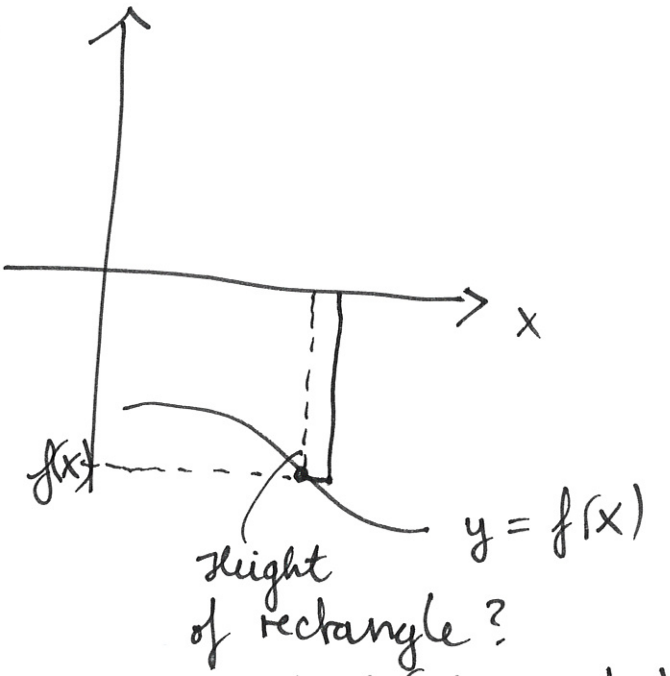
So (*) comes from that $\underbrace{I_1}_{A_1} + \underbrace{I_2}_{-A_2} = 4 + (-4) = 0$

When $f(x) \leq 0$ in $[a, b]$:

Area between the x-axis and the graph of $y = f(x)$ is:

$$A = \int_a^b - f(x)\,dx$$

So: $\displaystyle\int_a^b f(x)\,dx = -A$



$0 - f(x) = -f(x)$

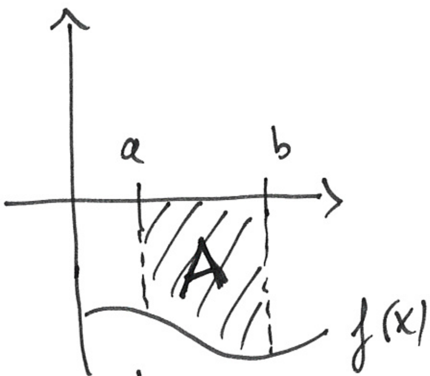
o—o

Different cases:

i) $f(x) \geq 0$:

$A = \int_a^b f(x)\,dx$

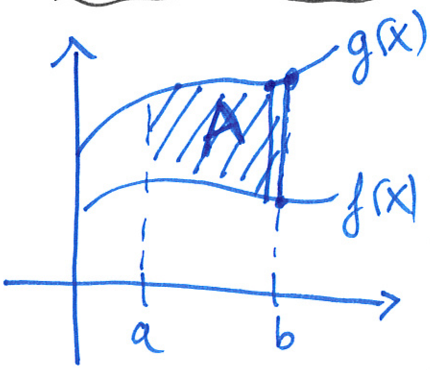
ii) $f(x) \leq 0$:



$A = \int_a^b - f(x)\,dx$

$-A = \int_a^b f(x)\,dx$

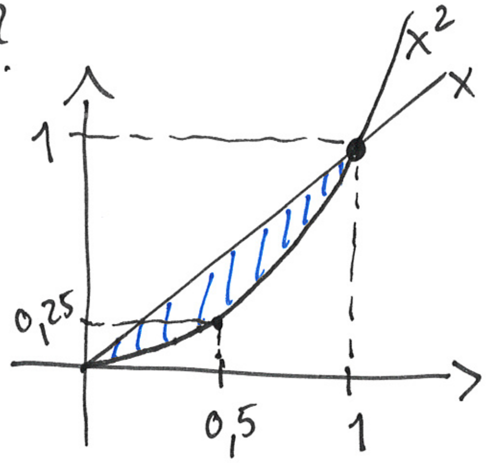
iii) $f(x) \leq g(x)$



$A = \int_a^b g(x) - f(x)\,dx$

Ex: What is the area between $y = x$ and $y = x^2$ in $[0,1]$?

IMPORTANT: Make figure!

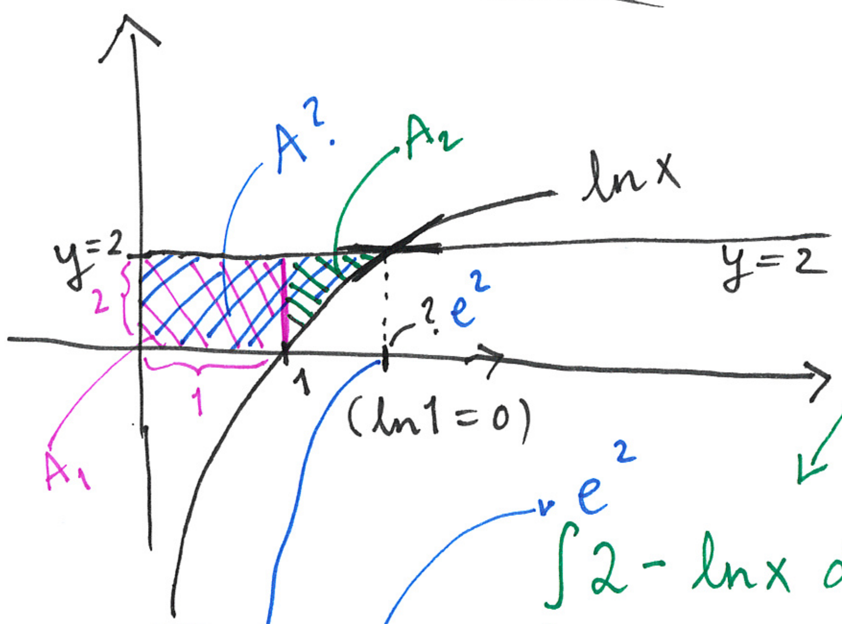


$$A = \int_0^1 x - x^2 \, dx = \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_{x=0}^{1}$$

$$= \frac{1}{2} 1^2 - \frac{1}{3} 1^3 - \underbrace{\left(\frac{1}{2} 0^2 - \frac{1}{3} 0^3 \right)}_{0}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \underline{\underline{\frac{1}{6}}} \left(\approx 0{,}167 \right)$$

Ex: What is the area bounded by $y = \ln x$, $y = 2$, y-axis and the x-axis?



Split A into two:

$A_1 = 1 \cdot 2 = 2$

$A_2 = ?$

$A = A_1 + A_2$

$$\int_1^{e^2} 2 - \ln x \, dx$$

$\ln x = 2$

$e^{\ln x} = e^2$

$x = e^2$

$$A = A_1 + A_2 = 2 + \int_1^{e^2} 2 - \underbrace{\ln x}_{1 \cdot \ln x \text{; int. by parts}} \, dx$$

$$= 2 + \left[2x - (x \ln x - x) \right]_{x=1}^{e^2}$$

$$= 2 + \left[3x - x \ln x \right]_{x=1}^{e^2}$$

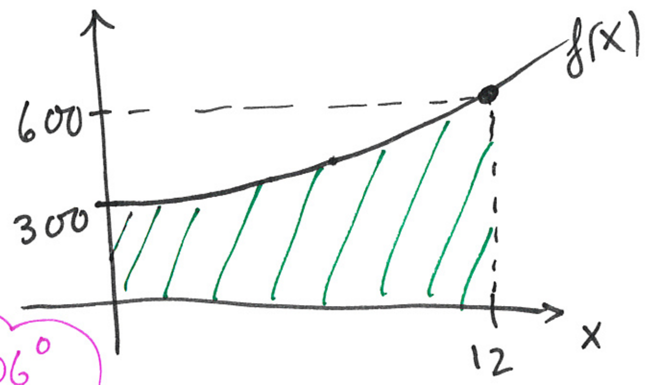
$$= 2 + (3e^2 - e^2 \underbrace{\ln e^2}_{2}) - (3 \cdot 1 - 1 \cdot \underbrace{\ln 1}_{0})$$

$$= 2 + 3e^2 - 2e^2 - 3 = e^2 - 1 \; (\approx 6{,}389)$$

Economic applications of the definite integral

a) Continuous cash flows:

Ex: $f(x) = 300 \cdot 1{,}06^x$ (cash flow in MNOK/year)



$300 \cdot 1{,}06^0 = 300$

Total cash flow in 12 years? The area under the graph in $[0, 12]$.

"Rule of 72":
Doubling takes approx. $\frac{72}{6} = 12$ years when smth. grows with 6% per year

$$= \int_0^{12} f(x)\,dx = \int_0^{12} 300\cdot 1{,}06^x\,dx$$

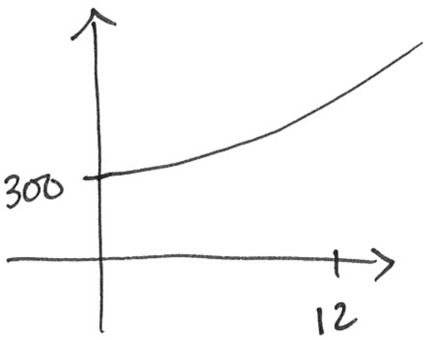
Int. rule from 1st lecture

$$= \left[300\cdot \frac{1{,}06^x}{\ln(1{,}06)}\right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1{,}06)}\left[1{,}06^x\right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1{,}06)}\left(1{,}06^{12}-1\right) \approx 5{,}211 \text{ MNOK}$$

$\overbrace{\text{Net present value}}^{\boxed{\text{NPV}}}$ of a continuous cash flow



$f(x) = 300\cdot 1{,}06^x$, cash flow

Assume: r = discount rate = 10%; continuous discounting.

NPV: $$\int_0^{12} f(x)\,e^{-rx}\,dx = \int_0^{12} 300\cdot 1{,}06^x\cdot e^{-0{,}1x}\,dx$$

$$= 300\int_0^{12} 1{,}06^x e^{-0{,}1x}\,dx$$

$$= 300\int_0^{12} e^{\ln(1{,}06)x}\, e^{-0{,}1x}\,dx$$

TRICK!

$$e^{\boxed{\ln(1{,}06)x}} = \left(e^{\ln 1{,}06}\right)^x = 1{,}06^x$$

$$\left(a^{nm} = (a^n)^m\right)$$

$$= 300 \int_0^{12} e^{(\ln(1{,}06) - 0{,}1)x} \, dx$$

Substitution:

$u = (\ln(1{,}06) - 0{,}1)x$

$du = (\ln(1{,}06) - 0{,}1)\,dx$

$dx = \dfrac{1}{\ln(1{,}06) - 0{,}1}\, du$

$$= 300 \left[\frac{1}{\ln(1{,}06) - 0{,}1} \cdot e^{(\ln(1{,}06) - 0{,}1)x} \right]_{x=0}^{12}$$

$$= \frac{300}{\ln(1{,}06) - 0{,}1} \left(e^{(\ln(1{,}06) - 0{,}1)\,12} - \underbrace{e^{0}}_{1} \right)$$

$$\approx \underline{2{,}832 \text{ MNOK}}$$

FORMULAS: (Economic applications of the definite integral)

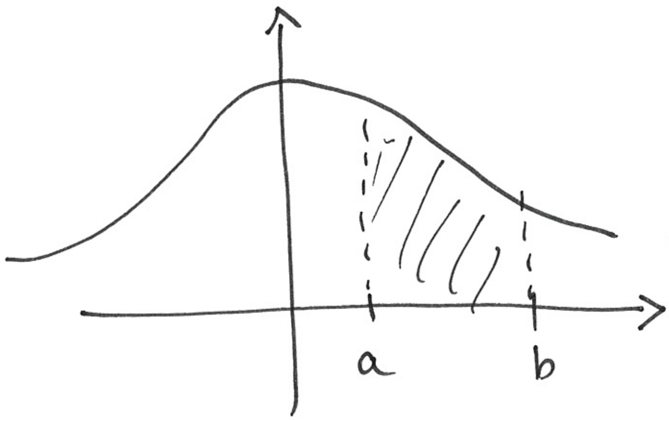
Total cash flow: $\displaystyle\int_0^T \underbrace{f(x)}_{\text{cash flow per time unit}} dx$

NPV of cash flow: $\displaystyle\int_0^T f(x)\, e^{-rx}\, dx$

discount rate

Other applications

Probabilities (continuous random variables):



$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, called the std. normal probability distribution, $X \sim N(0, 1)$

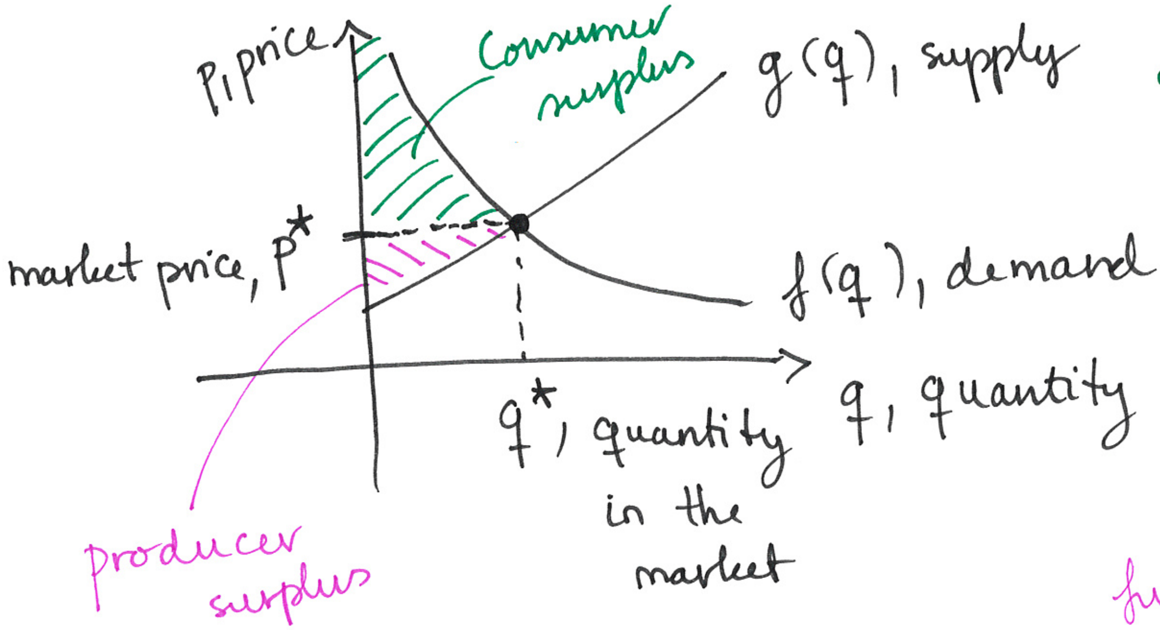
Ex: Height, foot sizes, temperature, stock prices — at a particular time

random variable

$$\underbrace{P(a \leq X \leq b)}_{\text{Prob. that the cont. random variable } X \text{ takes a value between } a \text{ and } b.} = \int_a^b f(x)\,dx$$

Consumer / producer surplus

$\boxed{CS}$ / $\boxed{PS}$



- $p = f(q)$, demand function (inverse)
- $p = g(q)$, supply func. (inverse)

CS : Consumer surplus :

$$CS = \int_0^{q^*} f(q) - p^* \, dq$$

PS : Producer surplus :

$$PS = \int_0^{q^*} p^* - g(q) \, dq$$

In the setting above

Ex: $f(q) = \dfrac{100}{q+5}$, $g(q) = q+5$

What is market price p^*?

$$f(q) = g(q)$$

$$\frac{100}{q+5} = q+5$$

$$100 = (q+5)^2$$

since prices must be non-neg.

$$q+5 = \pm\sqrt{100} = \pm 10 = 10$$

$$q = 5$$

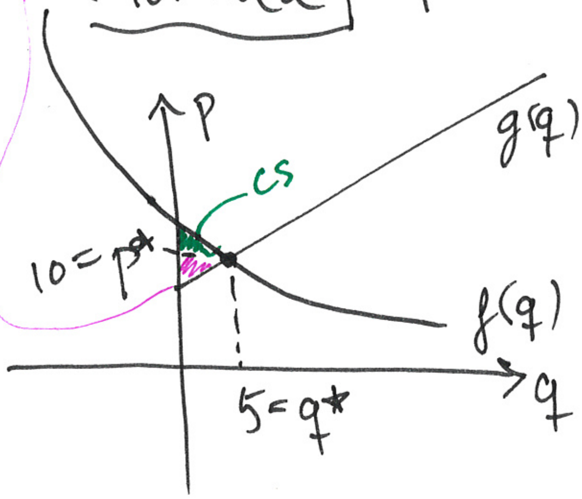
$$\underline{q^* = 5} \rightsquigarrow p^* = g(q^*) = q^* + 5$$

$$= \underline{10}$$

$$PS = \int_0^5 \underbrace{10}_{p^*} - g(q)\,dq = \int_0^5 10 - (q+5)\,dq$$

Formula

$$= \int_0^5 5 - q\,dq = \ldots = \underline{12{,}5}$$



CS = ...

Recap exercise:

Compute the definite integral:

$$\int_2^3 x\sqrt{x^2+1}\,dx$$

Substitution:

Plan?

$u = x^2 + 1$

Pressure pt: Remember to change the int. bounds.

$x = 2 \Rightarrow u = 2^2 + 1 = 5$

$x = 3 \Rightarrow u = 3^2 + 1 = 10$