Recap last time

EBA 1180
Lecture 27
Spring 23

Integration methods

- Substitution: $u = \underbrace{g(x)}_{\text{ugly!}}$

  int. version of "Chain rule"

  $$du = g'(x)\,dx$$

  $$dx = \frac{1}{g'(x)}\,du \qquad (g'(x) \neq 0)$$

- Integration by parts: $\int u' v \, dx = uv - \int u v' \, dx$

  Int. version of "product rule"

Int. by parts ctd.

Ex:

$$\int \underbrace{x}_{u'} \underbrace{\ln x}_{v} \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x} \, dx$$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$u' = x \Rightarrow u = \frac{1}{2}x^2$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Ex: $\int 2x\, e^x\, dx = e^x \cdot 2x - \int e^x \cdot 2\, dx$

$u' = e^x \Rightarrow u = e^x$

$v = 2x \Rightarrow v' = 2$

$$= 2xe^x - 2\int e^x\, dx$$

$$= 2xe^x - 2e^x + C$$

Opposite roles?

$$\int 2x\, e^x\, dx = x^2 e^x - \underbrace{\int x^2 e^x\, dx}_{\text{Looks worse!}}$$

$u' = 2x \Rightarrow u = x^2$

$v = e^x \Rightarrow v' = e^x$

Ex: $\int \ln x\, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v}\, dx$

TRICK

not too bad when anti-diff'ed

nice when diff'ed

$u' = 1 \Rightarrow u = x$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$$= x \ln x - \int \cancel{x} \frac{1}{\cancel{x}}\, dx$$

$$= x \ln x - x + C$$

$$\boxed{\text{FORMULA: } \int \ln x \, dx = x \ln x - x + C}$$

Ex: $\int \underbrace{x^2}_{v} \underbrace{e^x}_{u'} dx = e^x x^2 - \int e^x 2x \, dx$

PLAN? 2 * integration by parts

$u' = e^x \Rightarrow u = e^x$

$v = x^2 \Rightarrow v' = 2x$

$= e^x x^2 - \int 2x e^x \, dx$

$= \underline{\underline{e^x x^2 - 2xe^x + 2e^x + C}}$

See prev. pg.

Int. by parts

In general:

$\int x^k e^x \, dx \rightsquigarrow$ k times integration by parts

o—o

Problem set 27

1.) MET 1180 Spring 2017

$f(x) = 0{,}6 \ln(1+x) + 0{,}4 \ln(1-x), \quad 0 \leq x < 1$

a) Find max. pt., x^*, and the maximal value $f(x^*)$:

PLAN: Differentiate & set equal 0 $\Rightarrow$ Candidate pt!
Check whether max. pt.

Problem set 27

1) MET 1180 Spring 17

Exercise 1: $f(x) = 0{,}6 \ln(1+x) + 0{,}4 \ln(1-x)$, $0 \le x < 1$

a) Find max. pt. x^* & maximal value: $f(x^*)$

$$f'(x) = 0{,}6 \frac{1}{1+x} + 0{,}4 \frac{1}{1-x} \underbrace{(-1)}_{\text{diff'ate the core}}$$

maximize/minimize func.: Differentiate and set equal 0

$$= \frac{0{,}6}{1+x} - \frac{0{,}4}{1-x}$$

$$= \frac{0{,}6(1-x) - 0{,}4(1+x)}{(1+x)(1-x)}$$

Common denominator

$$= \frac{0{,}6 - 0{,}6x - 0{,}4 - 0{,}4x}{(1+x)(1-x)}$$

$$= \frac{0{,}2 - x}{(1+x)(1-x)}$$

To do so: Need to find the sign of the derivative

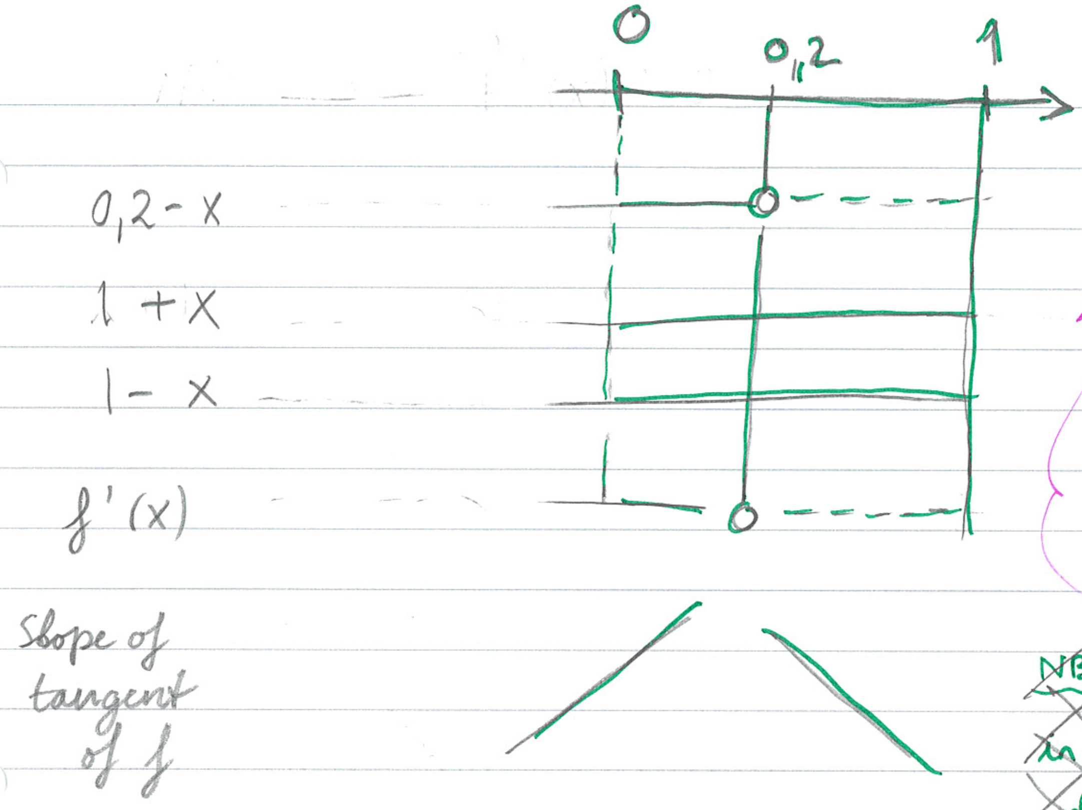
So: $f'(x) = 0$ gives

$$\frac{0{,}2 - x}{(1+x)(1-x)} = 0$$

$$0{,}2 - x = 0$$

$$\underline{x = 0{,}2}$$

Candidate for the max. pt: Need to check whether actually max



Cap the sign diagram here because f only defined for x < 1

~~NB: f only defined in green part of diagram (could erase grey bit)~~

From this, we see that our critical (candidate) point $x = 0,2$ is in fact a global max. point, so

$$x^* = 0,2.$$

where the func. is defined

The maximum value of f is:

$$f(x^*) = f(0,2) = 0,6 \ln(1,2) + 0,4 \ln(0,8)$$

$$\approx 0,0201$$

b) Determine whether f is convex or concave.

$$f''(x) = \left(\frac{0,2-x}{1+x} \cdot \frac{1}{1-x}\right)'$$

$$= \frac{(1+x)(-1) - (0,2-x)\cdot 1}{(1+x)^2} \cdot \frac{1}{1-x}$$

$$+ \frac{0,2-x}{1+x} \cdot \frac{1}{(1-x)^2} (-1)^2$$

quotient rule combined with product rule

Forgot this in lecture (from chain rule or power rule): messed everything up.

$$= \frac{(-1-\cancel{x}-0{,}2+\cancel{x})(1-x)+(0{,}2-x)(1+x)}{(1-x)^2(1+x)^2}$$

Get common denominator

$$= \frac{-1{,}2+1{,}2x+0{,}2+0{,}2x-x-x^2}{(1-x)^2(1+x)^2}$$

$$= \frac{-1+0{,}4x-x^2}{(1-x)^2(1+x)^2}$$

$$-x^2+0{,}4x-1=0 \quad |\cdot(-5)$$

Determine sign of nominator: When is it 0?

Multiply by 5 to get rid of 0,4

$$5x^2-2x+5=0$$

abc-formula: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{2\pm\sqrt{4-4\cdot5\cdot5}}{2\cdot5} = \frac{2\pm\sqrt{4-100}}{10}$$

Negative

$\Rightarrow$ No real roots!

So $x^2-0{,}4x-1$ is never 0.

Is it positive or negative?

Insert some point in D_f

$$x=0 \Rightarrow x^2-0{,}4x-1=0^2-0{,}4\cdot0-1=-1<0$$

Negative!

Also $(1-x)^2\geq 0$ and $(1+x)^2\geq 0$, hence

$$\left(f''(x) = \frac{\text{neg.}}{\text{pos.} \cdot \text{pos.}} = \text{neg.} \right)$$

$f''(x) < 0$ for all x where it's defined.

Hence, f is concave.

c) Show that $f(x) < 0$ when $x > 2x^*$:

→ See sign diagram

From a), $f'(x) < 0$ for $x > x^* = 0,2$

Hence, f is decreasing for $x > x^* = 0,2$

$f' < 0 \Leftrightarrow f$ decreasing

Furthermore,

$$f(2x^*) = f(0,4) = 0,6 \ln(1,4) + 0,4 \ln(0,6)$$
$$\approx -0,0024 < 0$$

Hence, $f(\overbrace{2x^*}^{0,4}) < 0$ and $f(x)$ decreases for $x > \overbrace{x^*}^{0,2}$, in particular for $x > \overbrace{2x^*}^{0,4}$ $(> 0,2 = x^*)$

Therefore, $f(x) < 0$ when $x > 2x^*$.

because starts at smth. neg. in. $2x^*$ and decreases from there.

d) Sketch the graph of f:

We know: • $f(\overbrace{0,2}^{x^*}) \approx 0,0201$ From a)

• $f(\underbrace{0,4}_{2x^*}) \approx -0,0024$ From b)

• f is increasing for $x < 0,2$ and decreasing for $x > 0,2$. From a)

From sign scheme of the derivative

• f is concave From b)

• f is defined on $[0,1)$. From exercise text

Where does f start?

$f(0)$?

$$f(0) = 0,6 \overbrace{\ln(1+0)}^{0} + 0,4 \overbrace{\ln(1-0)}^{0}$$

$$= 0$$

What happens when we approach 1?

$\lim_{x \to 1} f(x)$?

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} 0,6 \underbrace{\ln(1+x)}_{\to 0,6\ln(2)} + 0,4 \underbrace{\ln(1-x)}_{\to -\infty}$$

$1+x \to 2$, $1-x \to 0$

since $\lim_{y \to 0} \ln(y) = -\infty$

$$= -\infty$$

"$0,6 \ln 2 + (-\infty) = -\infty$"

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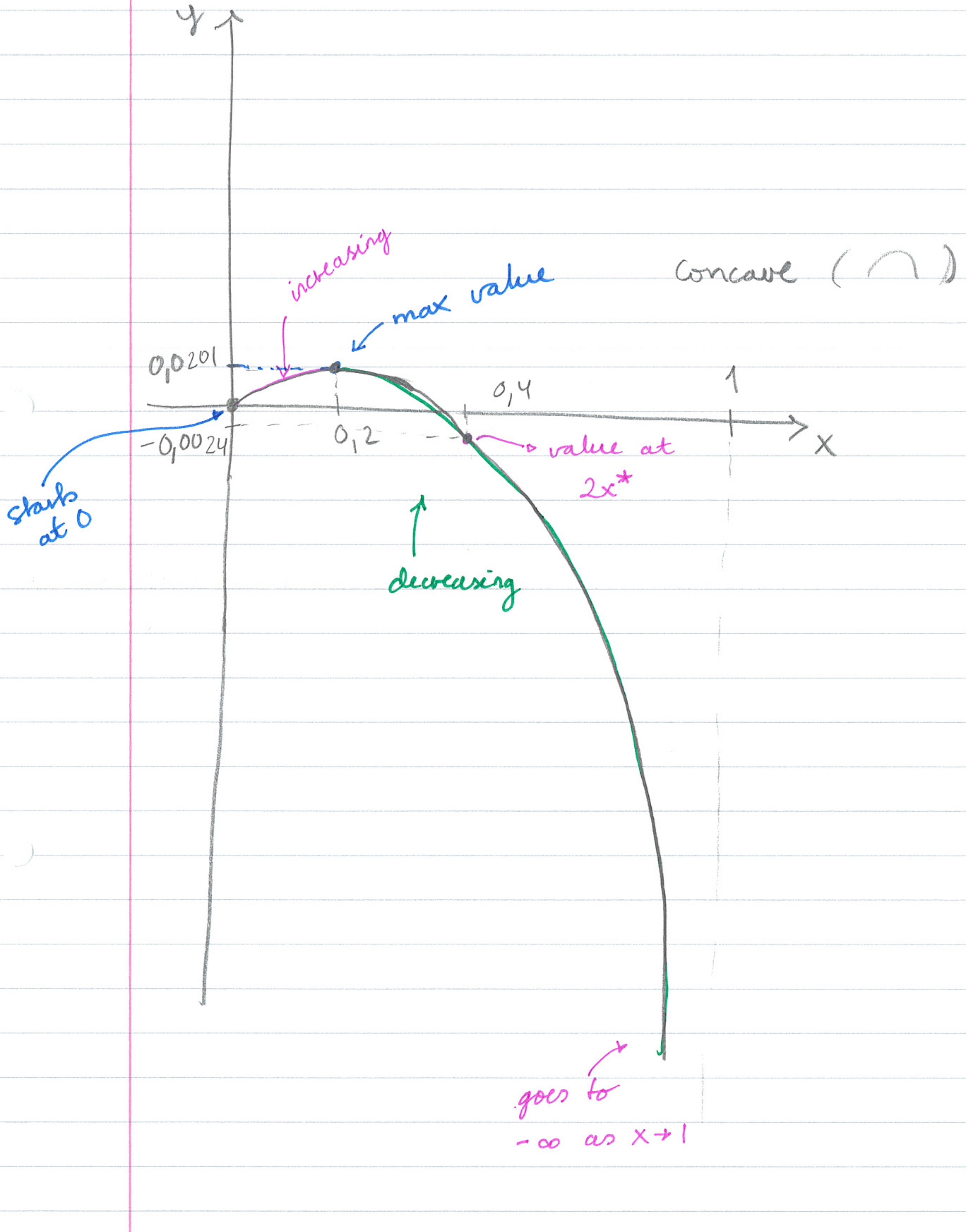
Draw all of this!



2) MET 11803 Fall 2018

$$f(x) = \frac{e^{1-\sqrt{x}}}{\sqrt{x}}, \quad x > 0$$

a) $f'(x) = ?$

$$f'(x) = \frac{\left(e^{1-\sqrt{x}}\right)' \cdot \sqrt{x} - e^{1-\sqrt{x}} \frac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

Quotient-rule

$$= \frac{\left(e^{1-\sqrt{x}}\right)' \sqrt{x} - \frac{e^{1-\sqrt{x}}}{2\sqrt{x}}}{x}$$

$$= \frac{\overbrace{e^{1-\sqrt{x}}}^{g'(u)} \overbrace{\left(-\frac{1}{2\cancel{\sqrt{x}}}\right)}^{u'} \cancel{\sqrt{x}} - \frac{1}{2\sqrt{x}} e^{1-\sqrt{x}}}{x}$$

Chain rule:

$u = 1 - \sqrt{x}$

$u' = -\frac{1}{2\sqrt{x}}$

Recall: $\sqrt{x} = x^{\frac{1}{2}}$, then regular rule for diff'ing powers

$$= \frac{e^{1-\sqrt{x}}\left(-\sqrt{x} - 1\right)}{2x\sqrt{x}}$$

$$\underbrace{e^{u}}_{g(u)} \Rightarrow g'(u) = e^{u}$$

b) Show that f is decreasing in $D_f = (0, \infty)$:

Since f decreasing $\Leftrightarrow f' < 0$

Suffices to show that $f'(x) < 0$ for $x \in (0, \infty)$.

element in

Consider the expression in a) for f': When $x > 0$, we see that

Denominator: $\overbrace{\overbrace{x}^{+}\overbrace{\sqrt{x}}^{+}}^{+} > 0 \Rightarrow$ Positive denominator

Nominator: • $e^{\overbrace{u}^{+}} > 0$ for all u, in particular $\underbrace{e^{1-\sqrt{x}}}_{+} > 0$ for all $x > 0$.

• $\underbrace{-\underbrace{\sqrt{x}}_{+}}_{-} \underbrace{-1}_{-\,(-1)} < 0$ for $x > 0 \Rightarrow$ Negative nominator.

Hence, $f'(x) < 0$ for all $x \in (0, \infty)$, so f i decreasing in D_f.

$f' = \frac{\text{neg.}}{\text{pos.}} = \text{neg.}$

c) Determine the limits

$$\lim_{x\to 0^+} f(x) \text{ and } \lim_{x\to\infty} f(x):$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1-\sqrt{x}}}{\sqrt{x}}$$

$e^{1-\sqrt{x}} \to e^{1-\sqrt{0}} = e^1 = e$

$\sqrt{x} \to \sqrt{0^+} = 0^+$

$\to$ "$\frac{e}{0^+}$", $+\infty$

$= \underline{\underline{\infty}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{1-\sqrt{x}}}{\sqrt{x}}$$

$e^{1-\sqrt{x}} \to$ "$e^{1-\sqrt{\infty}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$"

$\sqrt{x} \to$ "$\sqrt{\infty} = \infty$"

$\to$ "$\frac{0}{\infty}$" $= 0$

$= \underline{\underline{0}}$

d) Sketch the graph based on what we have found out and mark the area between the graph of f and the x-axis (for $x > 0$) in the sketch:

We know: • f is decreasing on $(0,\infty) = D_f$ b)

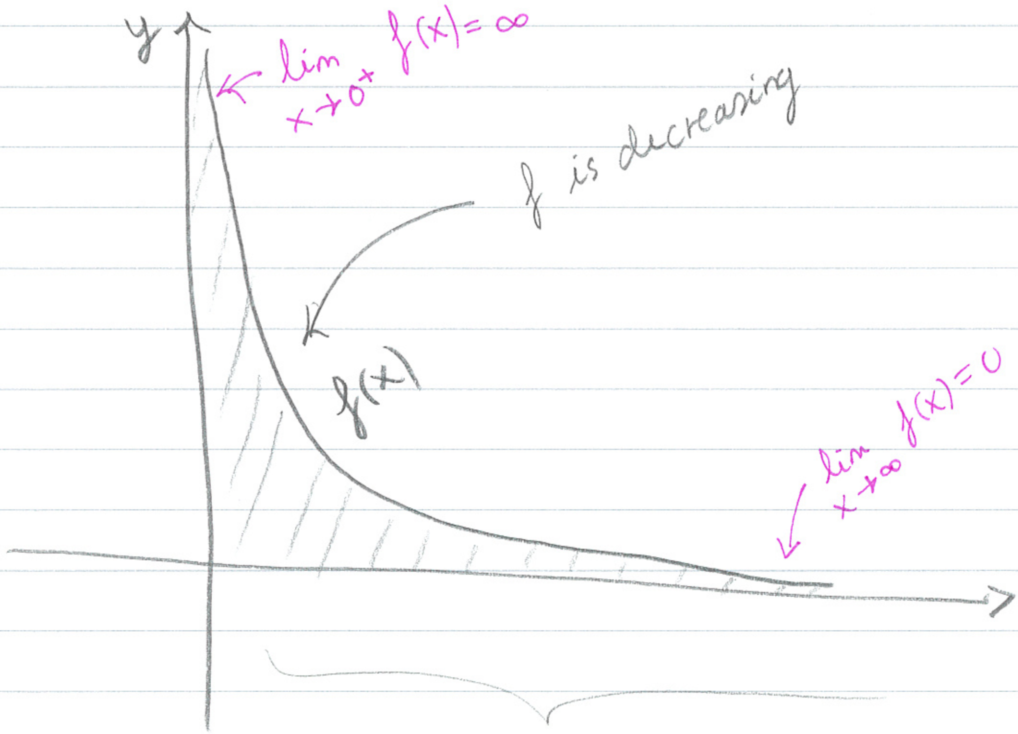
So f is only defined for $x \in (0,\infty)$

• $\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to \infty} f(x) = 0$

From c)

Let's draw these facts!



f is defined for $x \in (0,\infty)$.

The area in question is marked.