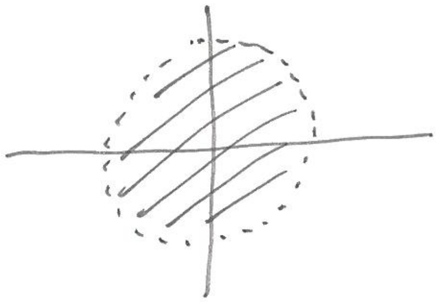
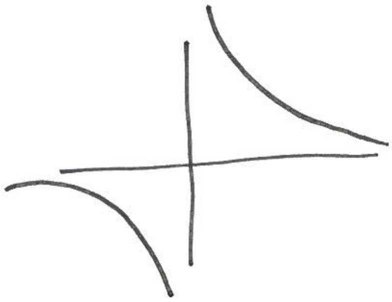
Problem Sheet 2
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Sheet 2

1.





$S = \{(x,y) : x^2+y^2<1\}$

$\partial S = \{(x,y) : x^2+y^2=1\}$

open, not closed

bounded (incl. in $B(\underline{0},2)$)

not compact

$T = \{(x,y) : xy=1\}$

$\partial T = \{(x,y) : xy=1\}$

not open, closed ($\partial T \subseteq T$)

not bounded (for any $M>0$, there are points in T not in $B(\underline{0},M)$ since $\lim_{x\to 0} 1/x = \pm\infty$)

not compact

> Note: In $\mathbb{R}^n$, a set cannot be open and closed (except $\emptyset$, $\mathbb{R}^n$). This is not in general true for topological spaces.

2. $\lim x_n = \lim n+1/n = \infty$ not convergent, not bounded

3. $\|x+y\|^2 = (x+y)\cdot(x+y) = x\cdot x + 2\, x\cdot y + y\cdot y \leq x\cdot x + 2\,|x\cdot y| + y\cdot y$

$\leq x\cdot x + 2\,\|x\|\cdot\|y\| + y\cdot y = \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2$

(this inequality in Cauchy-Schwartz' inequality)

$\Downarrow$

$\|x+y\|^2 \leq (\|x\|+\|y\|)^2$

$\Updownarrow$

$\|x+y\| \leq \|x\|+\|y\|$

4. Check that $f \cdot g = \int_0^1 f(t)g(t)dt$ defines inner prod. on $C(I, \mathbb{R})$:

i) $f \cdot g = g \cdot f$: $\quad f \cdot g = \int_0^1 f(t)g(t)dt = \int_0^1 g(t)f(t)\,dt = g \cdot f$ ok.

ii) $(af+bg)\cdot h = a\cdot(f\cdot h) + b\cdot(g \cdot h)$:

$$(af+bg)\cdot h = \int_0^1 (af(t)+bg(t))\,h(t)\,dt$$
$$= \int_0^1 a\cdot f(t)h(t) + b\,g(t)h(t)\,dt$$
$$= a\cdot\int_0^1 f(t)h(t)dt + b\int_0^1 g(t)h(t)dt = a(f\cdot h) + b(g\cdot h)$$ ok.

iii) $f\cdot f \geq 0$: $f \cdot f = \int_0^1 f(t)^2 dt \geq 0$ since $f(t)^2 \geq 0$ for all $t \in [0,1]$ ok.

$f\cdot f = 0 \Rightarrow f = 0$: $\quad f\cdot f = \int_0^1 f(t)^2 dt = 0 \Rightarrow f(t) = 0$ for all $t\in[0,1]$ (since f is continuous)

$\Rightarrow f = 0$ ok.

$$t^2 \cdot t^3 = \int_0^1 t^2 t^3 dt = \tfrac{1}{6}t^6\Big|_0^1 = \underline{\tfrac{1}{6}}$$

$$d(t^2,t^3) = \|t^2 - t^3\| = \left((t^2-t^3)\cdot(t^2-t^3)\right)^{1/2} = \sqrt{\int_0^1 (t^2-t^3)(t^2-t^3)\,dt}$$

$$= \sqrt{\left|\tfrac{1}{5}t^5 - 2\cdot\tfrac{1}{6}t^6 + \tfrac{1}{7}t^7\right|_0^1} = \sqrt{\tfrac{1}{5} - \tfrac{1}{3} + \tfrac{1}{7}} = \underline{\sqrt{\tfrac{1}{105}}}$$

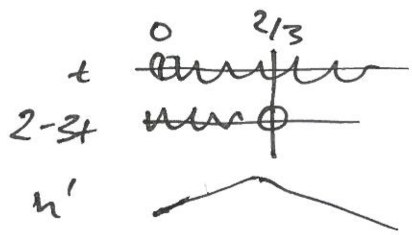
5. $$\|t^2 - t^3\| = d(t^2,t^3) = \sup_{t\in[0,1]} |t^2-t^3| = \sup_{t\in[0,1]} (t^2-t^3) = \underline{4/27}$$

Let $h(t) = t^2 - t^3$, $t \in [0,1]$

$h'(t) = 2t - 3t^2$

$= t(2-3t) = 0$

$t = 0$, $t = 2/3$



max for $h(t)$ on $[0,1]$ is at $t = 2/3$

$\Downarrow$

$h = (2/3)^2 - (2/3)^3 = \frac{4}{9} - \frac{8}{27} = \frac{12-8}{27}$

$= \underline{4/27}$

6. $f_n = t^n$

Compute $d(f_n, f_{n+k})$: ← If (f_n) is Cauchy, then for any fixed n big enough, $d(f_n, f_m)$ is very small for all $m \geq n$. Set $k = m-n$, and compute $d(f_n, f_{n+k})$.

$$\sup_{t\in[0,1]} |t^n - t^{n+k}| =$$

$$\sup_{t\in[0,1]} (t^n - t^{n+k}) = \frac{k}{n+k} \cdot \left(\frac{n}{n+k}\right)^{n/k}$$

Set $h(t) = t^n - t^{n+k}$ on $[0,1]$

$$h'(t) = nt^{n-1} - (n+k)t^{n+k-1} = t^{n-1}\left(n - (n+k)t^k\right) = 0$$

$$t = 0, \quad t = \left(\frac{n}{n+k}\right)^{1/k}$$

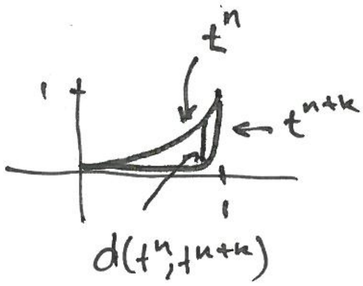
Max for h: (on $[0,1]$) $\quad t = \left(\frac{n}{n+k}\right)^{1/k}$

$$h = \left(\frac{n}{n+k}\right)^{n/k} - \left(\frac{n}{n+k}\right)^{\frac{n+k}{k}}$$

$$= \left(\frac{n}{n+k}\right)^{n/k}\left(1 - \frac{n}{n+k}\right) = \frac{k}{n+k} \cdot \left(\frac{n}{n+k}\right)^{n/k}$$

What happens with $d(t^n, t^{n+k})$ when n is large and fixed and k varies (and $k \to \infty$)?

$$\lim_{k\to\infty} \frac{k}{n+k} \cdot \left(\frac{n}{n+k}\right)^{n/k} = 1$$



Proof: $\displaystyle\lim_{k\to\infty} \frac{k}{n+k} = \lim_{k\to\infty} \frac{1}{\frac{n}{k}+1} = 1$

$$\ln\left(\lim_{k\to\infty}\left(\frac{n}{n+k}\right)^{n/k}\right) = \lim \ln\left(\frac{n}{n+k}\right)^{n/k} = \lim \frac{n}{k}\cdot\ln\left(\frac{n}{n+k}\right)$$

$$= \lim_{k\to\infty} \frac{n\ln(n/n+k)}{k} = \lim \frac{n \cdot \frac{n+k}{n} \cdot (-1) \cdot n/(n+k)^2}{1} = -\lim_{k\to\infty} \frac{n}{n+k} = 0$$

$$\Rightarrow \lim \left(\frac{n}{n+k}\right)^{n/k} = e^0 = 1$$

This means that (f_n) is <u>not Cauchy</u>; if it was, then

$$\lim_{n\to\infty} d(f_n, f_{n+k}) = 0$$

for any n big enough. (Recall defn. of Cauchy: For any $\varepsilon > 0$, there is N s.t. $n \geq N$ implies $d(f_n, f_m) < \varepsilon$ for all $n, m > N$. If $\varepsilon = 1/2$, this is not possible; no matter how big N is, when $m = N+k$ and $k \to \infty$, the metric $d(f_n, f_{n+k}) \to 1$.)

<u>Alternative argument:</u> Using completeness (Lecture 3)

If (f_n) is Cauchy, it converges to some $f \in C([0,1], \mathbb{R})$ since $C([0,1], \mathbb{R})$ is complete. This means that for all $\varepsilon > 0$

$$d(f_n, f) < \varepsilon \quad \text{for any } n > N$$

But $d(f_n, f) = \sup_{t \in [0,1]} |f_n(t) - f(t)| < \varepsilon$, so in particular $|f_n(t^*) - f(t^*)| < \varepsilon$ for any fixed $t^* \in [0,1]$. This means that

$$f(t) = \lim_{n\to\infty} f_n(t) \quad \text{for any } t \in [0,1] \quad \text{(pointwise convergence)}$$

But if $t \in [0,1)$, then $\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} t^n = 0$
$t = 1$, then $\lim f_n(t) = \lim_{n\to\infty} 1^n = 1$

so

$$f(t) = \begin{cases} 0, & t \in [0,1) \\ 1, & t = 1 \end{cases}$$

This is not continuous, a contradiction.

Hence (f_n) is <u>not Cauchy</u>.