Solutions Problem Sheet I:

1. a) $\begin{vmatrix} 2-\lambda & -3 \\ 7 & -8-\lambda \end{vmatrix} = \lambda^2+6\lambda+5=0$

$\lambda_1=-1,\ \lambda_2=-5$

$\lambda=-1$:

$\begin{pmatrix} 3 & -3 \\ 7 & -7 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda=-5$:

$\begin{pmatrix} 7 & -3 \\ 7 & -3 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t\cdot\begin{pmatrix} 3 \\ 7 \end{pmatrix}$

b) $\begin{vmatrix} 1-\lambda & 3 & 0 \\ 2 & -\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(\lambda^2-\lambda-6)=0$

$\lambda_1=2,\ \lambda_2=3,\ \lambda_3=-2$

$\lambda=2$:

$\begin{pmatrix} -1 & 3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} \underline{x} = \underline{0}$

$\downarrow$

$\begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda=3$:

$\begin{pmatrix} -2 & 3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} \underline{x} = \underline{0}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\lambda=-2$:

$\begin{pmatrix} 3 & 3 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 4 \end{pmatrix} \underline{x} = \underline{0}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t\cdot\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

c) $\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\underline{\lambda_1 = \lambda_2 = 3}$

$\underline{\lambda = 3}$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x} = \underline{0}$

$\Rightarrow \underline{x} = \underline{t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

2. a) i) $|A| = (-1)(-5) = \underline{5} \neq 0 \Rightarrow \text{rk } A = \underline{2}$

ii) $|A| = 2 \cdot 3 \cdot (-2) = \underline{-12} \neq 0 \Rightarrow \text{rk } A = \underline{3}$

iii) $|A| = 3 \cdot 3 = \underline{9} \neq 0 \Rightarrow \text{rk } A = \underline{2}$

b) A is positive / neg. defin / indefinite depending on the quadr. form $\underline{x}^T A \underline{x}$. This gives: Symmetric matrix A', and

cannot use eigenval. of A since A not symm.

i) $A' = \begin{pmatrix} 2 & 2 \\ 2 & -8 \end{pmatrix}$ negative definite $\Rightarrow$ A indefinite
$(|A| = -16 - 4 < 0)$

ii) $A' = \begin{pmatrix} 1 & 2.5 & 0.5 \\ 2.5 & 0 & -0.5 \\ 0.5 & -0.5 & 2 \end{pmatrix}$ ~~negative~~ indefinite $\Rightarrow$ A indefinite
$(D_1 = 1 > 0, D_2 = -2.5^2 < 0)$

iii) $A' = \begin{pmatrix} 3 & 0.5 \\ 0.5 & 3 \end{pmatrix}$ positive definite $\Rightarrow$ A positive definite
$(D_1 = 3, D_2 = 9 - 0.5^2 > 0)$

c) i) Diagonalizable: $D = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix}$

ii) Diagonalizable: $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $P = \begin{pmatrix} 0 & 3 & -2 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

iii) Not diagonalizable: $\lambda_1 = \lambda_2 = 3$ (mult. 2), only one lin. indep. eigenvector

3. a) $\left.\begin{array}{l} D_1 = 2 > 0 \\ D_2 = 8 - 1 = 7 > 0 \\ D_3 = |A| = 6 > 0 \end{array}\right\}$ A positive definite

b) $\left.\begin{array}{l} D_1 = 1 > 0 \\ D_2 = 1 - 4 = -3 < 0 \\ \vdots \end{array}\right\}$ A indefinite

4\. a) $\underline{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ gives $\underline{x}_{t+1} = A \cdot \underline{x}_t$, $A = \begin{pmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{pmatrix}$

$\underline{x}_t = A^t \cdot \underline{x}_0$ for all t

Alt I: Eigenvalue/-vector for A

$$\begin{vmatrix} 0.75-\lambda & 0.35 \\ 0.25 & 0.65-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.4 = 0$$

$$\lambda_1 = 1,\ \lambda_2 = 0.4$$

$\lambda = 1$: $\begin{pmatrix} -0.25 & 0.35 \\ 0.25 & -0.35 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

$\lambda = 0.4$: $\begin{pmatrix} 0.35 & 0.35 \\ 0.25 & 0.25 \end{pmatrix} \underline{x} = \underline{0} \Rightarrow \underline{x} = t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$A^t = (PDP^{-1})^t = PD^tP^{-1} = \begin{pmatrix} 7 & 1 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1^t & 0 \\ 0 & 0.4^t \end{pmatrix} \cdot \frac{1}{(-12)} \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix}$$

$\downarrow$ as $t \to \infty$

$$\begin{pmatrix} 7 & 1 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{-12} \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 5 & 0 \end{pmatrix} \frac{1}{(-12)} \cdot \begin{pmatrix} -1 & -1 \\ -5 & 7 \end{pmatrix} = \frac{1}{-12} \cdot \begin{pmatrix} -7 & -7 \\ -5 & -5 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 7 & 7 \\ 5 & 5 \end{pmatrix}$$

$$\underline{x}_t = A^t \underline{x}_0 \to \frac{1}{12} \begin{pmatrix} 7 & 7 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 7x_0 + 7y_0 \\ 5x_0 + 5y_0 \end{pmatrix} = \begin{pmatrix} 7/12 \\ 5/12 \end{pmatrix}$$

Alt. 2:

$$x_0 = c_1 \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow A^t x_0 = c_1 A^t \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 A^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= c_1 \cdot 1^t \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} + c_2 \cdot (0.4)^t \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\to c_1 \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Since $x_0 + y_0 = 1$, we have

$$c_1 \cdot 7 + c_1 \cdot 5 = 1 \Rightarrow c_1 = 1/12 \Rightarrow A^t x_0 \to \begin{pmatrix} 7/12 \\ 5/12 \end{pmatrix}$$