Exam	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 2nd, 2018 at 0900 - 1200

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

(a) Solve the linear system $(A+I)^2 \cdot \mathbf{v} = \mathbf{0}$, where I is the identity matrix, and show that not all solutions $\mathbf{v}$ of the linear system are eigenvectors of A.
(b) Show that A is not diagonalizable.
(c) A vector $\mathbf{v}$ is called a *generalized eigenvector* for A if $(A - \lambda I)^n \cdot \mathbf{v} = \mathbf{0}$ for some real number λ and some integer $n \geq 1$. Explain that any eigenvector for A is a generalized eigenvector, and find 3 generalized eigenvectors of A that are linearly independent.

Question 2.

We consider the function $f(x, y) = \sqrt{x} \cdot y$ defined on the domain $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$.

(a) Explain that D is a convex set, and determine whether f is a concave function on D.
(b) Find the set $f(D) = \{f(x, y) : (x, y) \in D\}$ of attainable values for f. Is $f(D)$ compact?
(c) Determine the values of a such that $U_f(a) = \{(x, y) \in D : f(x, y) \geq a\}$ is a convex set.
(d) Solve the constrained optimization problem

$$\min f(x, y) = \sqrt{x} \cdot y \text{ when } x^2 + y^2 \leq 2x$$

It can be useful to sketch the set $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2x\}$ of admissible points, and the level curve $f(x, y) = c$ passing through the minimizer.

Question 3.

Let A be an $n \times n$-matrix that is non-negative and with unit column sums; that is, such that $a_{ij} \geq 0$ for all i, j and such that $a_{1j} + a_{2j} + \cdots + a_{nj} = 1$ for each j. We consider

$$\Delta_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, x_2, \ldots, x_n \geq 0,\ x_1 + x_2 + \cdots + x_n = 1\} \subseteq \mathbb{R}^n$$

as a set of column vectors $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^T$.

(a) Show that the left multiplication $\mathbf{x} \mapsto A \cdot \mathbf{x}$ defines a well-defined function $A : \Delta_{n-1} \to \Delta_{n-1}$.
(b) Use Brouwer's fixed point theorem to show that the map $A : \Delta_{n-1} \to \Delta_{n-1}$ has a fixed point. You may use, without proof, that A is a continuous function.
(c) Find the fixed points of $A : \Delta_2 \to \Delta_2$ when A is the matrix given by

$$A = \begin{pmatrix} 0 & 0.6 & 0.5 \\ 0.7 & 0.4 & 0.5 \\ 0.3 & 0 & 0 \end{pmatrix}$$