LECTURE 9

21.09.2020

KAROLINE MOE

DRE 7017

1. OPTIMAL CONTROL THEORY – CURRENT VALUE

2. DISCRETE TIME DYNAMIC OPTIMIZATION PROBLEMS
   FINITE HORIZON

3. INFINITE HORIZON

OPTIMAL CONTROL THEORY – CURRENT VALUE FORMULATION

Optimal control with discount factor e^{-rt}

$$\max \int_{t_0}^{t_1} f(t,x,u)e^{-rt}\,dt \quad \text{when} \quad \begin{aligned} &x(t_0)=x_0 \\ &\dot{x}' = g(t,x,u) \\ &u \in U \subseteq \mathbb{R} \\ &a)\ x(t_1)=x_1 \\ &b)\ x(t_1)\ \text{free} \end{aligned}$$

Rewrite ordinary Hamiltonian

$$H = p_0 f(t,x,u)e^{-rt} + pg(t,x,u)$$

$$\begin{aligned} H^c = He^{rt} &= p_0 f(t,x,u) + pe^{rt} g(t,x,u) \\ &= \lambda_0 f(t,x,u) + \lambda g(t,x,u) \end{aligned}$$

NOTE:

$$\lambda = pe^{rt}, \text{ so } \lambda' = p'e^{rt} + rpe^{rt}$$

$$= p'e^{rt} + r\lambda$$

$$\underline{p' = e^{-rt}(\lambda' - r\lambda)}$$

compare

and $\underline{\dfrac{\partial H^c}{\partial x} = \dfrac{\partial H}{\partial x} \cdot e^{rt}}$ so $p' = -\dfrac{\partial H}{\partial x} = \underline{-\dfrac{\partial H^c}{\partial x} e^{-rt}}$

MAX PRINCIPLE – CURRENT VALUE

(A) $u = u^*(t)$ maximizes $H^c(t,x,u)$ $\left(\dfrac{\partial H^c}{\partial u} = 0\right)$

(B) $\lambda' - r\lambda = -\dfrac{\partial H^c}{\partial x}$

(C) Transversatility conditions
a) $\lambda(t_1)$ no cond. b) $\lambda(t_1) = 0$

Sufficient if $\lambda_0 = 1$ and H^c concave in (x,u)

EX: $\max \int_0^{20} (4K - u^2) e^{-0{,}25t}$ $\qquad \dot{K} = -0{,}25K + u$

$r = \frac{1}{4}$ easier to work with fractions

$K(0) = K_0$

$K(20)$ is free

$u \in [0; \infty)$

Current value Hamiltonian

$$H^C = 4K - u^2 + \lambda\left(-\tfrac{1}{4}K + u\right) \qquad (\lambda_0 = 1)$$

concave in (K, u) as sum of concave

Ⓐ $\dfrac{\partial H^C}{\partial u} = -2u + \lambda$

(Comes from a practical situation, so must have $u^*(t) > 0$.
Assuming $u^*(t) > 0$ (checked later))

$$\underline{u^*(t) = \tfrac{1}{2}\lambda}$$

Ⓑ $\dot{\lambda} - \frac{1}{4}\lambda = -\dfrac{\partial H}{\partial x} = -4 + \frac{1}{4}\lambda$

$\dot{\lambda} = \frac{1}{2}\lambda - 4$ $\qquad$ s.s. $\lambda_e = 8$

"eigenvalue" $\frac{1}{2} - x = 0$

"eigenvector" 1

} Analogous to the case with systems of lin diff eq.

$$\underline{\lambda = Ce^{\frac{1}{2}t} + 8}$$

Transv. cond

$\lambda(20) = 0$ since $K(t_1)$ free

implies $\lambda(20) = Ce^{10} + 8 = 0$

$C = -8e^{-10}$

$\lambda = 8(1 - e^{-10 + \frac{1}{2}t})$

$\underline{u^*(t) = \frac{1}{2} \cdot \lambda = 4(1 - e^{-10 + \frac{1}{2}t})}$ $\quad (> 0 \text{ for } t \in [0, 20))$

Diff of K. eq

$$\dot{K} = -\tfrac{1}{4}K + 4(1 - e^{-10 + \frac{1}{2}t})$$

dep. of t, so not the simplest case. Easy to solve using integrating factor $e^{\frac{1}{4}t}$

$$\dot{K} + \tfrac{1}{4}K = 4(1 - e^{-10 + \frac{1}{2}t})$$

$$(e^{\frac{1}{4}t}K) = \int 4(1 - e^{-10 + \frac{1}{2}t}) e^{\frac{1}{4}t}\, dt$$

$$e^{\frac{1}{4}t}K = \int 4e^{\frac{1}{4}t} - 4e^{-10 + \frac{3}{4}t}\, dt$$

$$e^{\frac{1}{4}t}K = 16e^{\frac{1}{4}t} - \tfrac{16}{3}e^{-10 + \frac{3}{4}t} + D$$

$$K(t) = 16 - \tfrac{16}{3}e^{-10 + \frac{1}{2}t} + De^{-\frac{1}{4}t}$$

$K(0) = K_0 = 16 - \frac{16}{3}e^{-10} + D$

$D = K_0 - 16 + \frac{16}{3}e^{-10}$

$$\underline{K^*(t) = 16 - \tfrac{16}{3}e^{-10 + \frac{1}{2}t} + \left(K_0 - 16 + \tfrac{16}{3}e^{-10}\right)e^{-\frac{1}{4}t}}$$

DISCRETE TIME DYNAMIC OPTIMIZATION PROBLEMS
FINITE HORIZON

$$(*) \quad \underbrace{\max \sum_{t=0}^{T} f(t, x_t, u_t)}_{\text{objective function}} \quad \text{subject to} \quad \begin{array}{l} x_{t+1} = g(t, x_t, u_t) \\ x_0 \text{ given} \\ \underset{\text{control functions}}{u_t \in U} \end{array}$$

A choice of $u_0, .., u_T$ determines $x_0, .., x_T$

Admissible pairs $(\{x_t\}, \{u_t\})$

Optimal pair $(\{x_t^*\}, \{u_t^*\})$ maximizes $(*)$

DEF: Optimal value function at time s

$$J_s(x) = \max_{u_s, .., u_T \in U} \sum_{t=s}^{T} f(t, x_t, u_t) \quad \text{when} \quad \begin{array}{l} x_s = x \\ x_{t+1} = g(t, x_t, u_t) \\ \text{for } t > s \\ u_t \in U \end{array}$$

BELLMAN EQUATION (12.1.1 - FMEA)

$$(**) \quad J_s(x) = \max_{u \in U} \left[f(s, x, u) + J_{s+1}(g(s, x, u)) \right]$$

$$s = 0, 1, .., T-1$$

$$J_T(x) = \max_{u \in U} \{ f(T, x, u) \} \qquad (s = T)$$

(same holds with min everywhere - EXAM 2017 ③)

APPROACH - BACKWARDS

① Find $J_T(x)$ and $u_T^*(x)$

② Use $(**)$ to find $J_{T-1}(x)$ and $u_{T-1}^*(x)$.
And determine recursively all $J_T(x), .., J_0(x)$,
and $u_T^*(x), .., u_0^*(x)$.

③ Find $x_{t+1}^* = g(t, x_t^*, u_t^*)$ by choosing $u_0^*(x_0)$
to compute $x_1^* = g(t, x_0, u_0^*(x_0))$ and $u_1^*(x_1^*)$
to compute $x_2^* = g(t, x_1^*, u_1^*(x_1^*))$ and so on.

④ $J_0(x) = J_0(x_0)$

EX. 2 (FMEA 427)

$$\max \sum_{t=0}^{3} (1 + x_t - u_t^2)$$

$$\boxed{\begin{array}{c} x_{t+1} = x_t + u_t \\ (t = 0,1,2) \\ x_0 = 0 \\ u_t \in \mathbb{R} \end{array}}$$

$T = 3$ $\quad f(\cancel{t}, x, u) = 1 + x - u^2$ $\quad g(t, x, u) = x + u$

$J_3(x) = \max_u (1 + x - u^2)$ $\quad$ obtained for $u = 0$

$\underline{J_3(x) = 1 + x_3}$ $\quad \underline{u_3^* = 0}$

$$J_2(x) = \max_u \left(1 + x - u^2 + (1 + x_3)\right)$$
$$= \max_u (1 + x - u^2 + 1 + x + u)$$
$$= \max_u (2 + 2x + u - u^2)$$

$h_2(u) = 2 + 2x + u - u^2$

$h_2'(u) = 1 - 2u = 0$ $\quad h_2''(u) = -2 < 0$

$u = \frac{1}{2}$ $\quad$ so max

$$J_2(x) = 2 + 2x + \frac{1}{2} - \frac{1}{4}$$
$$\underline{= \frac{9}{4} + 2x}$$

$\underline{u_2^* = \frac{1}{2}}$

$$J_1(x) = \max_u \left(1 + x - u^2 + \frac{9}{4} + 2x_2\right)$$
$$= \max_u \left(\frac{13}{4} + x - u^2 + 2(x + u)\right)$$
$$= \max_u \left(\frac{13}{4} + 3x - u^2 + 2u\right)$$

$h_1(u) = \frac{13}{4} + 3x - u^2 + 2u$

$h_1'(u) = -2u + 2 = 0$ $\quad h_1''(u) = -2 < 0$

$u = 1$ $\quad$ so max

$$J_1(x) = \frac{13}{4} + 3x - 1 + 2$$
$$\underline{= \frac{17}{4} + 3x}$$

$\underline{u_1^* = 1}$

$$J_0(x) = \max_u \left(1 + x - u^2 + \frac{17}{4} + 3x_1\right)$$

$$= \max_u \left(1 + x - u^2 + \frac{17}{4} + 3(x+u)\right)$$

$$= \max_u \left(\frac{21}{4} + 4x - u^2 + 3u\right)$$

$$h_0(u) = \frac{21}{4} + 4x - u^2 + 3u$$

$$h_0'(u) = -2u + 3 = 0$$

$$u = \frac{3}{2}$$

$$J_0(x) = \frac{21}{4} + 4x - \left(\frac{3}{2}\right)^2 + 3 \cdot \frac{3}{2} \qquad \underline{u_0^* = \frac{3}{2}}$$

$$= \frac{21 - 9 + 18}{4} + 4x$$

$$= \frac{30}{4} + 4x$$

$$= \underline{\frac{15}{2} + 4x}$$

$u_0^* = \frac{3}{2}$ $\quad x_0^* = 0$

$u_1^* = 1$ $\quad x_1^* = \frac{3}{2}$

$u_2^* = \frac{1}{2}$ $\quad x_2^* = \frac{5}{2}$

$u_3^* = 0$ $\quad x_3^* = 3$

$$\underline{J_0(x_0) = J_0(0) = \frac{15}{2}}$$

NOTE: Since T is small, it is possible to solve this with Calculus:

$x_1 = x_0 + u_0 = u_0$

$x_2 = x_1 + u_1 = u_0 + u_1$

$x_3 = x_2 + u_2 = u_0 + u_1 + u_2$

Then $I = \sum_{t=0}^{3} (1 + x_t - u_t^2) = (1 - u_0^2) + (1 + u_0 - u_1^2) + (1 + u_0 + u_1 - u_2^2) + (1 + u_0 + u_1 + u_2 - u_3^2)$

$$= 4 + 3u_0 + 2u_1 + u_2 - u_0^2 - u_1^2 - u_2^2 - u_3^2$$

sum of concave functions, so stationary pt is max:

$\frac{\partial I}{\partial u_0} = 3 - 2u_0$ $\quad \frac{\partial I}{\partial u_1} = 2 - 2u_1$ $\quad \frac{\partial I}{\partial u_2} = 1 - 2u_2$ $\quad \frac{\partial I}{\partial u_3} = -2u_3$

$u_0^* = \frac{3}{2}$ $\quad u_1^* = 1$ $\quad u_2^* = \frac{1}{2}$ $\quad u_3^* = 0$

DISCRETE TIME DYNAMIC OPTIMIZATION PROBLEMS
INFINITE HORIZON

$$J(x) = \max \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \quad \text{when} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

- $\beta \in (0,1)$ discount factor
- $t \to \infty$
- f and g don't depend on t explicitly
- Assume $f(x_t, u_t)$ bounded, $|f(x_t,u_t)| < M$ for all t and some $M > 0$.
  This ensures finite sum

$$\sum_{t=0}^{\infty} \beta^t (f(x_t, u_t) \leq \sum_{t=0}^{\infty} \beta^t M \overset{\text{geom. series}}{=} \frac{M}{1-\beta} \leftarrow \text{finite}$$

BELLMAN EQUATION

$$(*) \quad J(x) = \max_{u \in U} \{ f(x,u) + \beta J(g(x,u)) \}$$

- Difficult to solve functional eq. for $J(x)$ (if not known)
- When ① $\beta \in (0,1)$ and ② f is bounded
  the Bellman equation has a <u>unique</u> bounded solution $J^*(x)$.
  $\Downarrow$
  Guess a function $J(x)$.
  If $(*)$ holds with $J(x)$, it is the unique solution.

- Proof:
  - $B(\mathbb{R}, \mathbb{R})$ is a complete metric space (sup norm)
  - $T: B(\mathbb{R},\mathbb{R}) \to B(\mathbb{R},\mathbb{R})$ is a contraction

    $$J \longmapsto x \mapsto \max_{u \in U} \{ f(x,u) + \beta J(g(x,u)) \}$$
  - Fixed pt theorem for contractions on complete metric spaces.

EX 12.3.1.

$$\max \sum_{t=0}^{\infty} \beta^t \left(-e^{-u_t} - \frac{1}{2} e^{-x_t}\right)$$

x_0 given
$x_{t+1} = 2x_t - u_t$
$U = \mathbb{R}$
$0 < \beta < 1$

Goal : Find $\alpha > 0$ s.t. $J(x) = -\alpha e^{-x}$ solves the Bellman eq. and show α unique.

LHS: $$J(x) = \max_{u \in U} \left(-e^{-u} - \frac{1}{2} e^{-x} + \beta J(2x-u)\right)$$

$$= \max_{u \in U} \left(-e^{-u} - \frac{1}{2} e^{-x} + \beta(-\alpha) e^{-(2x-u)}\right)$$

$$h(u) = -e^{-u} - \frac{1}{2} e^{-x} - \alpha\beta e^{-2x+u}$$

$$h'(u) = e^{-u} - \alpha\beta e^{-2x+u} = 0$$

$$e^{-u} = \alpha\beta e^{-2x} \cdot e^{u} \qquad \Big| \cdot e^{u} \cdot e^{2x} \cdot \frac{1}{\alpha\beta}$$

$$e^{2u} = \frac{1}{\alpha\beta} e^{2x}$$ (or just take logarithms)

$$\ln e^{2u} = \ln\left(\frac{1}{\alpha\beta}\right) + \ln e^{2x}$$ (lots of logarithms)

$$2u = -\ln(\alpha\beta) + 2x$$

$$\underline{u^* = -\frac{1}{2}\ln(\alpha\beta) + x}$$

$$J(x) = -e^{\frac{1}{2}\ln(\alpha\beta) - x} - \frac{1}{2} e^{-x} - \alpha\beta e^{-(2x + \frac{1}{2}\ln(\alpha\beta) - x)}$$

$$= -e^{\ln((\alpha\beta)^{\frac{1}{2}})} \cdot e^{-x} - \frac{1}{2} e^{-x} - \alpha\beta e^{-x} \cdot e^{-\ln((\alpha\beta)^{\frac{1}{2}})}$$

$$= e^{-x} \left(-(\alpha\beta)^{\frac{1}{2}} - \frac{1}{2} - (\alpha\beta)(\alpha\beta)^{-\frac{1}{2}}\right)$$

$$= e^{-x} \left(-\frac{1}{2} - 2(\alpha\beta)^{\frac{1}{2}}\right)$$

RHS $= -\alpha e^{-x}$, so:

$$\alpha = \frac{1}{2} + 2\sqrt{\alpha\beta}$$

$\alpha - \frac{1}{2} = 2(\alpha\beta)^{\frac{1}{2}}$ Let $z = \alpha^{\frac{1}{2}}$

$z^2 - 2\beta^{\frac{1}{2}} z - \frac{1}{2} = 0$

$$z = \frac{2\beta^{\frac{1}{2}} \pm \sqrt{4\beta - 4 \cdot \frac{-1}{2}}}{2}$$

$$\sqrt{\alpha} = \sqrt{\beta} \pm \frac{2\sqrt{\beta + \frac{1}{2}}}{2}$$

$\sqrt{\alpha} = \sqrt{\beta} \pm \sqrt{\beta + \frac{1}{2}}$, but since $\alpha > 0$ and $\sqrt{}$ increasing, $+$ is the only solution

$$\alpha = \beta + 2\sqrt{\beta^2 - \frac{1}{2}\beta} + \beta + \frac{1}{2}$$

$$= \underline{2\beta + 2\sqrt{\beta^2 - \frac{1}{2}\beta} + \frac{1}{2}}$$

$\underline{\underline{J(x) = -\left(\sqrt{\beta} + \sqrt{\beta + \frac{1}{2}}\right)^2 e^{-x}}}$ is the solution