LECTURE 6

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DRE 7017

(1) CONSTRAINED OPTIMIZATION

(2) NONLINEAR PROGRAMMING - INEQUALITY CONSTRAINTS

FMEA 3.3-3.6

ME 18-19

S 5-6, 7.7, 8.8.

① CONSTRAINED OPTIMIZATION $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$

$$\max\ (\min)\ f(x_1, \dots, x_n) \text{ subject to } \begin{cases} h_1(x_1,\dots,x_n) = a_1 \\ \quad \vdots \\ h_m(x_1,\dots,x_n) = a_m \end{cases}$$

$m < n$ (ensures freedom)

$\max\ (\min)\ f(\underline{x})$ and $h_j(\underline{x}) = b_j$ for $j = 1, \dots, m$.

objective function   m equality constraints

LAGRANGE'S METHOD – Two variables & one equality constraint

EX. (ME 18.4)

maximize $f(x_1, x_2) = x_1 x_2$

subject to $h(x_1, x_2) \equiv x_1 + 4x_2 = 16$.

THM (18.1. ME) – Two variables & one equality constraint

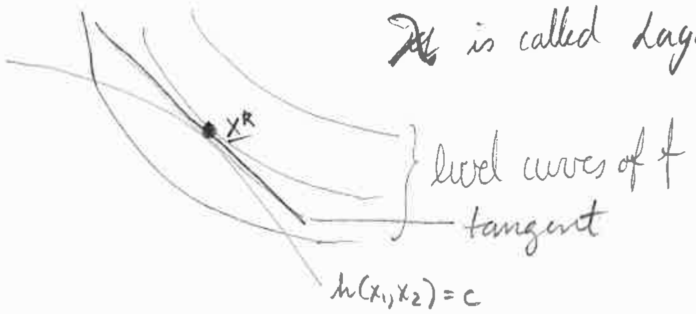
Let f and h be C^1 functions of two variables. Suppose that $\underline{x}^* = (x_1^*, x_2^*)$ is a solution of the problem maximize $f(x_1, x_2)$ subject to $h(x_1, x_2) = a$ and that $\underline{x}^*$ is not a critical point of h. (constraint qualification)

Then there is a $\lambda^* \in \mathbb{R}$ such that $(x_1^*, x_2^*, \lambda^*)$ is a critical point of the Lagrange function

$$\mathcal{L}(x_1, x_2, \mu) \equiv f(x_1, x_2) - \lambda\,(h(x_1, x_2) - a).$$

In other words $\frac{\partial \mathcal{L}}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = \frac{\partial \mathcal{L}}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = \frac{\partial \mathcal{L}}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) = 0$

λ is called Lagrange multiplier.



Slope of tangent:

$$-\frac{f'_{x_1}(\underline{x}^*)}{f'_{x_2}(\underline{x}^*)} = -\frac{h_{x_1}(\underline{x}^*)}{h_{x_2}(\underline{x}^*)}.$$

The highest level curve of f that touch $h(x_1, x_2) = c$ and the constraint set $h(x_1, x_2) = a$ have the same tangent at $\underline{x}^*$.

An unconstrained problem in one more variable!

EX. • Check constraint qualification

$h'_{x_1} = 1$

$h'_{x_2} = 4$, so always ok!

• $\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 - \lambda(x_1 + 4x_2 - 16)$

$$\text{FOC} \begin{cases} \mathcal{L}'_{x_1} = x_2 - \lambda = 0 \\ \mathcal{L}'_{x_2} = x_1 - 4\lambda = 0 \end{cases}$$

© $\mathcal{L}'_{\lambda} = x_1 + 4x_2 - 16 = 0$ same as eq. constraint $(x_1 + 4x_2 = 16)$

· Find all possible values $(x_1^*, x_2^*, \lambda^*)$ that satisfy the above

$\lambda = x_2$

$4\lambda = x_1$

$4\lambda + 4\lambda = 16$

$\underline{\lambda = 2}$

$\underline{x_1 = 8}$

$\underline{x_2 = 2}$

Usually not a linear system, but here it is!

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -4 & 0 \\ 1 & 4 & 0 & 16 \end{pmatrix}$$

(Unique solution)

$f(2,8) = 2 \cdot 8 = \underline{\underline{16}}$

$f(x_1,x_2) = x_1 x_2$ is a surface in $\mathbb{R}^3$

$h(x_1,x_2) \equiv x_1 + 4x_2 - 16$ is a plane in $\mathbb{R}^3$

f itself has a saddle pt at $(0,0)$

$f'_{x_1} = x_2$

$f'_{x_2} = x_1$

$f''_{x_1 x_1} = 0$

$f''_{x_1 x_2} = 1$

$f''_{x_2 x_2} = 0$

$H(f)(\underline{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$D_1 = 0$ $\quad \Delta_1 : 0 \quad 0$

$D_2 = -1$ $\quad \Delta_2 = -1$

so indefinite

LAGRANGE METHOD for n variables and ~~m~~ m eq. constraints

THM (18.2 ME)

Let $f, h_1, \ldots, h_m$ be C^1-functions ~~of~~ n variables.

maximizing $f(\underline{x})$ on $C_{\underline{h}} = \{\underline{x} \mid h_1(\underline{x}) = a_1, \ldots, h_m(\underline{x}) = a_m\}$

Suppose that $\underline{x}^* \in C_h$ and $\underline{x}^*$ (local) max or min on $C_{\underline{h}}$ and rank $D\underline{h}(\underline{x}^*) = m$, where

NDCQ

$$D\underline{h}(\underline{x}^*) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\underline{x}^*) & \cdots & \frac{\partial h_1}{\partial x_n}(\underline{x}^*) \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1}(\underline{x}^*) & \cdots & \frac{\partial h_m}{\partial x_n}(\underline{x}^*) \end{pmatrix}.$$

← Jacobian of $\underline{h}$
(rank $< m \Rightarrow \underline{x}^*$ critical pt. of $\underline{h}$)

Then there exist $\lambda_1^*, \ldots, \lambda_m^*$ s.t. $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*) = (\underline{x}^*, \underline{\lambda}^*)$

is a critical point of the Lagrangian

$$\mathcal{L}(\underline{x}, \underline{\lambda}) \equiv f(\underline{x}) - \lambda_1(h_1(\underline{x}) - a_1) - \ldots - \lambda_m(h_m(\underline{x}) - a_m),$$

i.e. $\frac{\partial \mathcal{L}}{\partial x_i}(\underline{x}^*, \underline{\lambda}^*) = 0 \quad \forall i = 1, \ldots, n$ FOC

$$\left\{ \begin{array}{l} \text{and } \frac{\partial \mathcal{L}}{\partial \lambda_j}(\underline{x}^*, \underline{\lambda}^*) = 0 \quad \forall j = 1, \ldots, m \\ \text{or equivalently } \quad h_j(\underline{x}^*) = a_j \quad \forall j = 1, \ldots, m \end{array} \right\} \quad C$$

SOLUTION METHOD

- Find all points with FOC + C statisfied

NDCQ - Find all points in D s.t. NDCQ is not satisfied.

If the Lagrange problem has a solution, then it has to be one of these points.

OBS: $\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 h_1(\underline{x}) - \ldots - \lambda_m h_m(\underline{x})$

gives exactly same

EX: (ME 18.7)

Maximize $f(x,y,z) = yz + xz$ subj. to $y^2+z^2=1$

$xz = 3$

- NDCQ: $h'_{1,x} = 0$ $\quad h'_{2,x} = z$

$h'_{1,y} = 2y$ $\quad h'_{2,y} = 0$

$h'_{1,z} = 2z$ $\quad h'_{2,z} = x$

$$\text{rank}\begin{pmatrix} 0 & 2y & 2z \\ z & 0 & x \end{pmatrix} = 2 \text{ if}$$

$(x,y,z) \neq (0,0,0)$
$(1,0,0)$
$(0,1,0)$

so $z^* \neq 0$ for all pts.

- $\mathcal{L}(x,y,z,\lambda,\mu) = yz + xz - \lambda(y^2+z^2-1) - \mu(xz-3)$

FOC: $\begin{cases} \mathcal{L}'_x = z - \mu z = 0 \\ \mathcal{L}'_y = z - 2y\lambda = 0 \\ \mathcal{L}'_z = y + x - 2z\lambda - \mu x = 0 \end{cases}$

C $\begin{cases} y^2+z^2=1 \\ xz = 3 \end{cases}$

- $z(1-\mu) = 0$

$z=0$	$\mu = 1$
$\begin{cases} 2y\lambda = 0 \\ y + x - \mu x = 0 \\ y^2 = 1 \\ 0 = 3 \Rightarrow \text{contr.} \end{cases}$	$\mathcal{L}'_z: y + \cancel{x} - 2z\lambda - \cancel{x} = 0$ $\lambda = \frac{y}{2z}$ $\mathcal{L}'_y: z - \frac{2y^2}{2z} = 0 \qquad z^2 = y^2$

$2y^2 = 1$

$y = \pm\frac{1}{\sqrt{2}} \qquad z = \pm\frac{1}{\sqrt{2}}$

$\lambda = \frac{1}{2}$

$\lambda = -\frac{1}{2}$

$x \cdot \left(\pm\frac{1}{\sqrt{2}}\right) = 3$

$x = \pm 3\sqrt{2}$

Critical pts:	$f(x,y,z)$
$(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 1)$	$3+\frac{1}{2}$ max
$(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}, 1)$	$3-\frac{1}{2}$ min
$(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 1)$	$3-\frac{1}{2}$ min
$(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, 1)$	$3+\frac{1}{2}$ max
$(0,0,0)$	0
$(1,0,0)$	0
$(0,1,0)$	0

$\left.\begin{matrix}0\\0\\0\end{matrix}\right\}$ min

SUFFICIENT CONDITIONS

* $D = \{\underline{x} \mid g_1(\underline{x}) = a_1, \ldots, g_m(\underline{x}) = a_m\} \subseteq \mathbb{R}^n$
  is a closed set.
  If D is bounded, it is compact, so $f(D)$ is compact and max/min $f(\underline{x})$ has solutions.

* second order conditions
  If $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC + C, then
  $\mathcal{L}(\underline{x}) = \mathcal{L}(\underline{x}, \underline{\lambda}^*)$ is concave $\Rightarrow \underline{x}^*$ is a maximizer
  $\mathcal{L}(\underline{x}, \lambda^*)$ is convex $\Rightarrow \underline{x}^*$ is a minimizer

INTERPRETATION OF LAGRANGE MULTIPLIER λ_i

Optimal value function:

$$f^*(\underline{a}) = \max_{(\min)} f(\underline{x}) \text{ when } \begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

(f profit, $\underline{a}$ resource vector)

Can rewrite: $\lambda_j(\underline{a}) = \dfrac{\partial f^*(\underline{a})}{\partial a_j}$ $\quad j = 1, \dots, m$

So λ_j tells us how the optimal value of the objective function changes wrt. a_j

Resource corresponding to j, λ_j shadow price

$\lambda_j > 0$ implies that an increase in a_j increases the optimal value function.

$\lambda_j < 0$ implies that an increase in a_j decreases the optimal value function

LINEAR APPROXIMATION

Small changes in a_j gives an effect on the optimal value function that can be approximated linearly.

$$\underbrace{f^*(\underline{a} + d\underline{a}) - f^*(\underline{a})}_{\text{difference on the optimal value fn.}} \approx \lambda_1(\underline{a})\, da_1 + \dots + \lambda_m(\underline{a})\, da_m$$

small change in a_1

compared to a_1

LOCAL SECOND-ORDER CONDITIONS

local max(min) $f(\underline{x})$ subject to $h_j(\underline{x}) = a_j$ $\quad j = 1, \dots, m$, $\quad m < n$

$$\mathcal{L}(\underline{x}) = f(\underline{x}) - \sum_{j=1}^{m} \lambda_j \left(h_j(\underline{x}) - a_j\right)$$

for $r = m+1, \dots, n$, define

$$B_r(x^*) = \begin{vmatrix} \mathbf{0} & \begin{matrix} \frac{\partial h_1(x^*)}{\partial x_1} & \cdots & \frac{\partial h_1(x^*)}{\partial x_r} \\ \vdots & & \vdots \\ \frac{\partial h_m(x^*)}{\partial x_1} & \cdots & \frac{\partial h_m(x^*)}{\partial x_r} \end{matrix} \\ \begin{matrix} \frac{\partial h_1(x^*)}{\partial x_1} & \cdots & \frac{\partial h_m(x^*)}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial h_1(x^*)}{\partial x_r} & \cdots & \frac{\partial h_m(x^*)}{\partial x_r} \end{matrix} & \underbrace{\begin{matrix} \mathcal{L}''_{11}(x^*) & \cdots & \mathcal{L}''_{1r}(x^*) \\ \vdots & & \vdots \\ \mathcal{L}''_{r1}(x^*) & \cdots & \mathcal{L}''_{rr}(x^*) \end{matrix}}_{r\text{th leading principal minor of } H(\mathcal{L})(\underline{x}^*)} \end{vmatrix}$$

r first partials of h_j (top rows)

r first partials of h_j (left columns)

THM (3.4.1 FMEA)

Suppose $f, h_1, \dots, h_m$ defined on $D \subseteq \mathbb{R}^n$,
$\underline{x}^*$ an interior pt in D, NDCQ ok,
and $f, h_1, \dots, h_m$ C^2 in $B(\underline{x}^*, \varepsilon)$. Then

(A) If $(-1)^m B_r(\underline{x}^*) > 0$ for $r = m+1, \dots, n$,
then $\underline{x}^*$ solves the local minimization pb.

(B) If $(-1)^r B_r(x^*) > 0$ for $r = m+1, \dots, n$,
then $\underline{x}^*$ solves the local maximization pb.

Not always fun to compute without CAS.

(2) NONLINEAR PROGRAMMING – INEQUALITY CONSTRAINTS

standard problem:

$$\max f(x_1, \dots, x_n) \text{ subject to } \begin{cases} h_1(x_1, \dots, x_n) \le a_1 \\ \vdots \\ h_m(x_1, \dots, x_n) \le a_m \end{cases}$$

- f, h_j C^1-functions.
- No longer restricted to $m < n$!
- minimizing $f(\underline{x})$ = maximizing $-f(\underline{x})$
- $h_j(\underline{x}) \ge a_j$ can be rewritten $-h_j(\underline{x}) \le -a_j$

KUHN-TUCKER CONDITIONS

$$\mathcal{L}(\underline{x}) = f(\underline{x}) - \lambda_1 h_1(\underline{x}) - \dots - \lambda_m h_m(\underline{x})$$

FOC	C	CSC - complementary slackness conditions
$\mathcal{L}'_{x_1} = 0$	$h_1(\underline{x}) \le a_1$	$\lambda_j \ge 0$, with $\lambda_j(h_j(\underline{x}) - a_j) = 0$
$\vdots$	$\vdots$	$j = 1, \dots, m.$
$\mathcal{L}'_{x_n} = 0$	$h_m(\underline{x}) \le a_m$	

(This is where all the fun sits...)
At least one of $\lambda_j \ge 0$ and $h_j(\underline{x}) \le a_j$ is $=$.

NDCQ in KUHN-TUCKER

- If $h_j(\underline{x}^*) = a_j$, the condition is called BINDING at $\underline{x}^*$ (or active)
- If $h_j(\underline{x}^*) < a_j$, — NONBINDING at $\underline{x}^*$. (or inactive)

NDCQ: rank J' is maximal, where J' is the matrix with rows $\nabla h_j(\underline{x}^*)$ for all binding conditions.

THM:

If $\underline{x}^*$ is a maximizer in a K-T-P in std form and NDCQ holds at $\underline{x}^*$, then there are $\lambda_1, \ldots, \lambda_m$ s.t. $(\underline{x}^*, \underline{\lambda})$ satisfy FOC+C+CSC.

SOLUTION METHOD: (As before)

① All points that satisfy FOC+C+CSC.

② All points in D that fail NDCQ.

If a max exists, it is one of these.

SUFFICIENT CONDITIONS

① D compact $\Updownarrow$ D bounded $\Rightarrow$ max/min $f(\underline{x})$ has a solution

second order cond. $\left\{ \begin{array}{l} \text{② If } (\underline{x}^*, \underline{\lambda}^*) \text{ satisfies FOC+C+CSC and} \\ \mathcal{L}(\underline{x}) = \mathcal{L}(\underline{x}, \lambda^*) \text{ is concave, then } \underline{x}^* \text{ is a maximizer} \end{array} \right.$

CSC - ADVISE

systematically go through all possible combinations:

- All constraints binding (1 case)
- One constraint non-binding (m cases)
- Two —"— ($\binom{m}{2}$ cases)

⋮

- All constraints non-binding (1 case)

EX. 3.5.1.

Solve $\max \underbrace{1-x^2-y^2}_{f(x,y)}$ subject to $\begin{matrix} x \geq 2 \\ y \geq 3 \end{matrix} \iff \begin{matrix} -x \leq -2 \\ -y \leq -3 \end{matrix}$

DIRECT ARGUMENT: $f(x,y)$ is max for $(x,y)=(2,3)$, since that is the smallest they can be – we subtract x^2 and y^2. And $f(2,3) = 1-2^2-3^2 = \underline{\underline{-12}}$

KUHN-TUCKER

$$\mathcal{L}(x,y,\lambda,\mu) = 1-x^2-y^2+\lambda x+\mu \cdot y$$

$$\text{FOC} \begin{cases} \mathcal{L}'_x = -2x+\lambda = 0 \\ \mathcal{L}'_y = -2y+\mu = 0 \end{cases}$$

C: $x \geq 2$, $y \geq 3$

CSC: $\lambda \geq 0$ with $\lambda(-x+2)=0$

$\mu \geq 0$ with $\mu(-y+3)=0$

FOC $\Rightarrow$ $\lambda = 2x > 0$, $\mu = 2y > 0$

so $x=2$, $y=3$, $\lambda=4$, $\mu=6$

$\mathcal{L}''_{xx} = -2$

$\mathcal{L}''_{xy} = 0$

$\mathcal{L}''_{yy} = -2$

$$H(\mathcal{L})(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$D_1 = -2 < 0$

$D_2 = 4 > 0$

so $H(\mathcal{L})(x,y)$ neg. definite, so $\mathcal{L}$ is concave, and $(2,3)$ is a max.

EXAMPLE

$$\max\ x^2+y^2 \quad \text{subject to } 2x^2+3y^2 \le 6$$

$$\mathcal{L} = x^2+y^2 - \lambda(2x^2+3y^2)$$

$$\mathcal{L}_x' = 2x - 4x\lambda = 0 = 2x(1-2\lambda)$$

$$\mathcal{L}_y' = 2y - 6y\lambda = 0 = 2y(1-3\lambda)$$

$C: \ 2x^2+3y^2 \le 6$

CSC. $\lambda \geq 0$ with $\lambda(2x^2+3y^2-6)=0$

- All binding: $\lambda \geq 0$ and $2x^2+3y^2-6=0$
  $\lambda = \frac{1}{2}$ and $y=0$, so $x = \pm\sqrt{3}$
  $\lambda = \frac{1}{3}$ and $x=0$, so $y = \pm\sqrt{2}$
  $\text{rk } f' = \text{rk}(4x \ \ 6y) = 1$ ok NDCQ

- No binding: $\lambda = 0$, $x=0$, $y=0$ shows up as stationary pt.

ADMISSIBLE PTS:

	$f(x,y)$	
$(\sqrt{3}, 0, \frac{1}{2})$	3	max
$(-\sqrt{3}, 0, \frac{1}{2})$	3	
$(0, \sqrt{2}, \frac{1}{3})$	2	
$(0, -\sqrt{2}, \frac{1}{3})$	2	
$(0, 0, 0)$	0	min

$\mathcal{L}_{xx}'' = 2-4\lambda$

$\mathcal{L}_{xy}'' = 0$

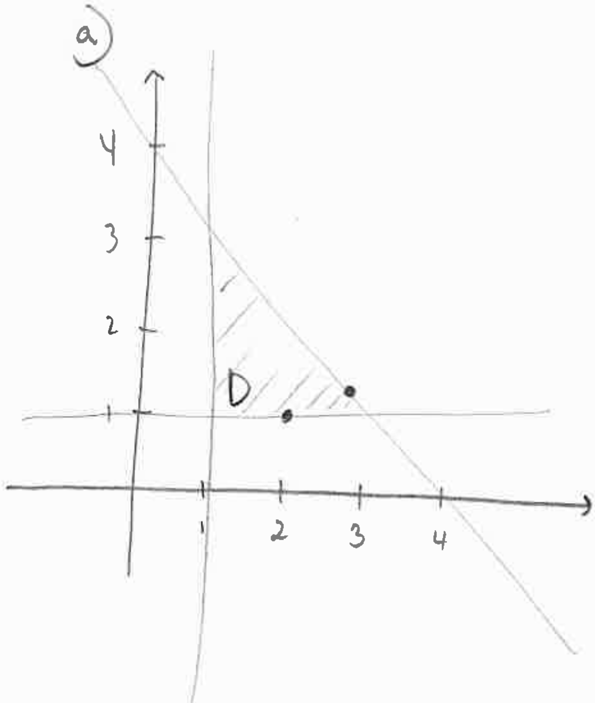
$\mathcal{L}_{yy}'' = 2-6\lambda$

$$H(\lambda) = \begin{pmatrix} 2-4\lambda & 0 \\ 0 & 2-6\lambda \end{pmatrix} \Rightarrow \begin{array}{l} D_1 = 2-4\lambda \qquad \Delta_1' = 2-6\lambda \\ D_2 = (2-4\lambda)(2-6\lambda) \end{array}$$

is neg. semidef for $\lambda = \frac{1}{2}$
is pos. semidef for $\lambda = \frac{1}{3}$
is pos. def for $\lambda = 0$ $\rightarrow$ convex.

PROBLEM max/min $\ln(x^2y) - x - y$ when $\begin{cases} x+y \le 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$ D

a)



D is closed and bounded. (compact)

$x+y \le 4$
$-x \le -1$
$-y \le -1$

b) $\mathcal{L}(x,y) = \ln(x^2y) - x - y - \lambda_1(x+y) + \lambda_2 x + \lambda_3 y$

FOC $\begin{cases} \mathcal{L}'_x = \dfrac{2xy}{x^2y} - 1 - \lambda_1 + \lambda_2 = \dfrac{2}{x} - 1 - \lambda_1 + \lambda_2 = 0 \\[2ex] \mathcal{L}'_y = \dfrac{x^2}{x^2y} - 1 - \lambda_1 + \lambda_3 = \dfrac{1}{y} - 1 - \lambda_1 + \lambda_3 = 0 \end{cases}$

C: $x+y \le 4$
$x \geq 1$
$y \geq 1$

CSC: $\lambda_1 \geq 0$ with $\lambda_1(4-(x+y)) = 0$
$\lambda_2 \geq 0$ with $\lambda_2(1-x) = 0$
$\lambda_3 \geq 0$ with $\lambda_3(1-y) = 0$

$\mathcal{L}''_{xx} = -\dfrac{2}{x^2}$

$\mathcal{L}''_{xy} = 0$

$\mathcal{L}''_{yy} = -\dfrac{1}{y^2}$

$H(\mathcal{L}) = \begin{pmatrix} -\frac{2}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$

$D_1 = -\dfrac{2}{x^2}$

$D_2 = \dfrac{2}{x^2y^2}$

so $H(\mathcal{L})$ neg. def.
$\mathcal{L}$ concave

① All cases ~~active~~ : binding

$$\left.\begin{array}{l} x+y=4 \\ x=1 \\ y=1 \end{array}\right\} \text{Impossible}$$

② Ⓐ $\lambda_1 = 0$ and $x=1, y=1$

$\frac{2}{1} - 1 + \lambda_2 = 0$

$\lambda_2 = -1 < 0$

contradiction

Ⓑ $\lambda_2 = 0$ $x+y=4$ and $\underline{y=1}$

$1 - 1 + \lambda_1 + \lambda_3 = 0$

$\lambda_1 = -\lambda_3$

only $\lambda_1 = \lambda_3 = 0$

But $x=3$ does not fullfil $\mathcal{L}_x' = 0$

Ⓒ $\lambda_3 = 0$ and $x+y=4$ and $x=1$

$y=3$

$\frac{1}{3} - 1 + \lambda_1 = 0$

$\lambda_1 = \frac{2}{3} > 0$

$\frac{2}{1} - 1 + \frac{2}{3} + \lambda_2 = 0$

$\frac{5}{3} + \lambda_2 = 0$

$\lambda_2 = -\frac{5}{3} < 0$

contradiction

③ Ⓐ $\lambda_1 = \lambda_2 = 0$ and $y=1$

$\lambda_3 = 0$ and $x=2$

$\underline{(2,1,0,0,0)}$ $rk(0,1)=1$, ok.

$-1,61 = f(2,1)$

Ⓑ $\lambda_1 = \lambda_3 = 0$ and $x=1$

$2 - 1 + \lambda_2 = 0$

$\lambda_2 = -1$, contradiction.

Ⓒ $\lambda_2 = \lambda_3 = 0$ $x+y=4$

$\lambda_1 = \frac{2}{x} - 1 = \frac{1}{y} - 1$

$2y = x$

$3y = 4$

$y = \frac{4}{3}$, $x = \frac{8}{3}$, $\lambda_1 = -\frac{1}{4} < 0$

Contradiction

④ $\lambda_1 = \lambda_2 = \lambda_3 = 0$

$x = 2$

$y = 1$

$(2,1,0,0,0)$

so $(2,1)$ maximizes $\ln(x^2y) - x - y$

on $x + y \leq 4$

$x \geq 1$

$y \geq 1$